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Modules over (qa)-rings

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Let R be a commutative ring with unit. When the total quotient ring Q of R is an Artinian ring we call R a (qa)-ring. In this paper we are mainly concerned with the theory of modules over such a ring. In §1, some preliminary results are summarized. In §2 we shall prove the following (Theorem 2. 10): Let R be a (qa)-ring with the self-injective total quotient ring, and let M be an h-divisible R-module such that $M/t(M)$ is an injective R-module. Then $t(M)$ is a direct summand. Some applications of the preceding result will be discussed in §3.

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1. Preliminaries

Let R be a commutative ring with 1 and let S be the set of all non zero-divisors in R. The total quotient ring R_S is denoted by Q, and K will denote the quotient module Q/R. Let M be a module (always assumed to be unitary) over the ring R. An element x in M is torsion if there is an element s in S such that $sx=0$, and torsion-free otherwise. M is called a torsion module if every element in M is torsion, and a torsion-free module if every element in M is trosion-free. Let M be an R-module. Then as is easily seen there is the unique maximal submodule which is torsion. This submodule will be denoted by $t(M)$ and will be called the torsion submodule of M. An R-module M is torsion-free if and only if $t(M)=0$.

Proposition 1.1. *Let M be an R-module. Then we have* $t(M)\cong \mathrm{Tor}_1^R(K, M)$.

Proof. From $0\to R\to Q\to K\to 0$, we have the following exact sequence: $0\to \mathrm{Tor}_1^R(Q, M)\to \mathrm{Tor}_1^R(K, M)\to M\to Q\otimes_R K$. But $\mathrm{Tor}_1^R(Q, M)=0$ since Q is a flat R-module, and by Proposition 1.4 $\mathrm{Tor}_1^R(K, M)$ is torsion. Thus $\mathrm{Tor}_1^R(K, M)\to t(M)$ is monomorphic. On the other hand, if N is a torsion-free module, then we have a canonical map: $N\to Q\otimes_R N$ is monomorphic. Therefore $\mathrm{Tor}_1^R(K, M)\to t(M)$ is an onto R-homomorphism. Thus $t(M)\cong \mathrm{Tor}_1^R(K, M)$.

Corollary 1. 2. *For any R-module M we have the following exact sequence:*

$$0\to M/t(M)\to Q\otimes_R M\to K\otimes_R M\to 0.$$