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On the Stable Homotopy Ring of Moore Spaces

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Introduction

Let p be a prime integer $\geqq 5$, $q=2(p-1)$, and $M_t=S^1 \cup_t e^2$ be a Moore space of type $(Z_t, 1)$. Denote by $\mathscr{A}_k(M_t)$ the stable track group $\{S^k M_t, M_t\}=$ $\operatorname{Dir}\lim\{[S^{n+k}M_t, S^n M_t], S\}$, S being the suspension functor. Then the direct sum $\mathscr{A}_*(M_{p^r})=\Sigma_k\mathscr{A}_k(M_{p^r})$ is an algebra over Z_{p^r} with the multiplication induced by the composition of maps. The structure of the ring $\mathscr{A}_*(M_{p^r})$ is studied by several authors [4] [6] [13] [14].

N. Yamamoto [14] has calculated the ring structure of $\mathscr{A}_*(M_p)$ for degree $<p^2q-4$, $q=2(p-1)$, from the results [8] on the stable homotopy ring $G_*=$ $\Sigma_k G_k$, $G_k=\operatorname{Dir}\lim \pi_{n+k}(S^n)$, of spheres. P. Hoffman [4] has introduced a differential in $\mathscr{A}_*(M_t)$ and studied the commutativity of the ring $\mathscr{A}_*(M_t)$ using this differential. H. Toda [13] has generalized Hoffman's results and obtained several useful relations involving the elements $\beta_{(t)}\in\mathscr{A}_{(tp+t-1)q-1}(M_p)$.

The purpose of this paper is to determine the ring structure of $\mathscr{A}_*(M_{p^r})$ for any $r\geqq 1$, within the limits of degree less than $(p^2+3p+1)q-6$.

Let $i\ (=i_r)$: $S^1\to M_{p^r}$ and $\pi\ (=\pi_r)$: $M_{p^r}\to S^2$ denote the natural maps and set $\delta\ (=\delta_r)=i\pi\in\mathscr{A}_{-1}(M_{p^r})$. We have in Proposition 2.3 a direct sum decomposition for odd t:

$$\mathscr{A}_k(M_t)\approx G_{k+1}\otimes Z_t+G_k\otimes Z_t+G_k*Z_t+G_{k-1}*Z_t.$$

Let $H\approx Z_{p^s}$ be a summand of G_k generated by an element γ. Then H gives summands Z_{p^m}, $Z_{p^m}+Z_{p^m}$ and Z_{p^m}, $m=\min\{r,s\}$, of $\mathscr{A}_{k+1}(M_{p^r})$, $\mathscr{A}_k(M_{p^r})$ and $\mathscr{A}_{k-1}(M_{p^r})$, via the above decomposition. In § 3, we construct elements $[\gamma]$ $(=[\gamma]_r)\in\mathscr{A}_{k+1}(M_{p^r})$ and $\langle\gamma\rangle\ (=\langle\gamma\rangle_r)\in\mathscr{A}_k(M_{p^r})$ for γ above, and we see in Lemma 3.3 that we can take the elements $[\gamma]$, $[\gamma]\delta$, $\langle\gamma\rangle$ and $\langle\gamma\rangle\delta$ for the generators of four cyclic summands of $\mathscr{A}_*(M_{p^r})$ given by H. Thus the additive structure of $\mathscr{A}_*(M_{p^r})$ is described by using such elements (Theorem 3.5).

In Propositions 3.8–3.9, we discuss the relations of the products $\langle\alpha\rangle[\beta]$ and $[\alpha][\beta]$ in $\mathscr{A}_*(M_{p^r})$ with the composition $\alpha\beta$ and the Toda bracket $\langle\alpha,$ $p^s,\beta\rangle$ in G_*. By these results and by employing the differential D (see (1.6) for the definition) in $\mathscr{A}_*(M_{p^r})$, we can calculate the ring structure of $\mathscr{A}_*(M_{p^r})$ from the results [5] [6] on G_*.