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Parabolic and Hyperbolic Infinite Networks

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Introduction

We shall classify in this paper a set of infinite networks into parabolic networks and hyperbolic networks of order p.

More precisely, let $N=\{X, Y, K, r\}$ be an infinite network which is connected and locally finite and which has no self-loop, let $\mathbf{D}^{(p)}(N)$ be the set of all real functions on X with finite Dirichlet integrals $D_p(u)$ of order p $(1<p\leqq\infty)$ and let $L_0(X)$ be the set of all real functions on X with finite supports. We say that N is of parabolic type of order p if there exists a nonempty finite subset A of X such that the value $d_p(A, \infty)$ of the following extremum problem $(*)$ on N relative to A and the ideal boundary ∞ of N vanishes:

$$(*)\qquad \text{Find}\quad d_p(A, \infty) = \inf\{D_p(u);\ u\in L_0(X) \text{ and } u = 1 \text{ on } A\}.$$

We say that N is of hyperbolic type of order p if it is not of parabolic type of order p.

We shall prove in § 3 that N is of parabolic type of order p $(1<p<\infty)$ if and only if any one of the following conditions is fulfilled: (C. 1) $1\in\mathbf{D}_0^{(p)}(N)$, (C. 2) $\mathbf{D}^{(p)}(N)=\mathbf{D}_0^{(p)}(N)$, where $\mathbf{D}_0^{(p)}(N)$ is the closure of $L_0(X)$ in $\mathbf{D}^{(p)}(N)$ with respect to the norm $\|u\|_p=[D_p(u)+|u(x_0)|^p]^{1/p}$ $(x_0\in X)$. Some practical criteria which assure that the network is of parabolic type of order p will be given in § 4 by means of some results in [4] and [6] concerning the extremal length of an infinite network.

In case $p=2$, this classification problem, which was partially studied by C. Blanc [2], is very analogous to the classification problem of Riemann surfaces (see for instance [1], [3] and [5]).

It will be shown in § 5 that if N is of parabolic type of order p_1 and if $1<p_1<p_2$, then N is of parabolic type of order p_2. By this fact, we define a parabolic index $\operatorname{ind} N$ of N as the infimum of $p>1$ for which N is of parabolic type of order p. Some geometric meaning of $\operatorname{ind} N$ will be shown by two examples.

§ 1. Some definitions related to an infinite network

Let X and Y be countable (infinite) sets and K be a function on $X\times Y$ satisfy-