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On the space of orderings and the group H

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Let F be a formally real field and P a preordering of F. In his paper [7], M. Marshall introduced an equivalence relation in the space $X(F/P)$ of orderings by making use of fans of index 8, and the notion of connected components of $X(F/P)$ by an equivalence class of the relation.

The main purpose of this paper is to show that the number of connected components of $X(F/P)$ coincides with the dimension of $\boldsymbol{Z}_2$-vector space $H(P)/P$ for a subgroup $H(P)$, which is defined in §2. We also show, in §3, that if $K=F(\sqrt{a})$ is a quadratic extension of F with a an element of Kaplansky's radical, then the number of connected components of $X(K/P')$ equals twice that of $X(F/P)$, where P' is the preordering $\Sigma P\cdot\dot{K}^2$ of K. We should note that the groups $H(P)$ and $H(P')$ are connected by an important relation $N^{-1}(H(P))=F\cdot H(P')$, where N is the norm map of K to F.

For a subset A in a set B, the cardinality of A will be denoted by $|A|$ and the complementary subset of A in B by $B-A$ or A^c.

§1. Preorderings and fans

Throughout this paper, a field F always means a formally real field. We denote by $\dot{F}$ the multiplicative group of F. For a multiplicative subgroup P of $\dot{F}$, P is said to be a preordering of F if P is additively closed and $\dot{F}^2\subseteq P$. We denote by $X(F)$ the space of all orderings σ of F and by $X(F/P)$ the subspace of all orderings σ with $P(\sigma)\supseteq P$, where $P(\sigma)$ is the positive cone of σ. For a subset Y of $X(F)$, we denote by $Y^{\perp}$ the preordering $\cap P(\sigma)$, $\sigma\in Y$. Conversely for any preordering P, there exists a subset $Y\subseteq X(F)$ such that $P=Y^{\perp}$. Thus we have $P=X(F/P)^{\perp}$ and in particular $X(F)^{\perp}=D_F(\infty)=\Sigma\dot{F}^2$. We put $\phi^{\perp}=\dot{F}$ for convenience. The topological structure of $X(F)$ is determined by Harrison sets $H(a)=\{\sigma\in X(F);\ a\in P(\sigma)\}$ as its subbasis, where a ranges over $\dot{F}$. An arbitrary open set in $X(F)$ is thus a union of sets of the form $H(a_1,\dots, a_r)=H(a_1)\cap\cdots\cap H(a_r)$. For a preordering P of F, we write $H(a_1,\dots, a_n/P)=H(a_1,\dots, a_n)\cap X(F/P)$ where $a_i\in\dot{F}$.

For two forms f and g over F, we write $f\sim g(\mathrm{mod}\,P)$ if for any $\sigma\in X(F/P)$, $sgn_\sigma(f)=sgn_\sigma(g)$ where $sgn_\sigma(f)$ and $sgn_\sigma(g)$ are the signatures at σ of f and g, respectively. If $f\sim g(\mathrm{mod}\,P)$ and $\dim f=\dim g$, we write $f\cong g(\mathrm{mod}\,P)$. For