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A NOTE ON MITCHELL'S FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS

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1. Introduction.

Let X be a nonempty compact convex subset of a Banach space, and let S be a semigroup of nonexpansive mappings of X info itself. Mitchell [3] proved that if S is left reversible, then X contains a common fixed point of S. In this note, we shall show that Mitchell's theorem can be further extended.

2. Fixed point theorem.

Let $C(X)$ be the Banach space of all continuous real functions on X, and let $B(S)$ be the Banach space of all bounded real functions on S. For each $x \in X$ and $f \in C(X)$, we define $x \otimes f \in B(S)$ by $(x \otimes f)(s) = f(sx)$. Let $L(S)$ be the linear span of $\{x \otimes f : x \in X, f \in C(X)\}$ in $B(S)$. A linear functional m on $L(S)$ is called a left invariant mean on $L(S)$ if $\|m\| = m(1) = 1$, $m(\varphi) \geqq 0$ whenever $\varphi \geqq 0$, and $m({}_s\varphi) = m(\varphi)$ for every $s \in S$ and $\varphi \in L(S)$ where ${}_s\varphi$ denotes the left translation of φ by s.

Takahashi [5] proved that if there exists a left invariant mean on $B(S)$, then X contains a common fixed point of S. The following is an extension of Takahashi's theorem.

THEOREM 1. *If there exists a left invariant mean on $L(S)$, then X contains a common fixed point of S.*

Proof. Let K be a minimal S-invariant nonempty compact convex subset of X, and let M be a minimal S-invariant nonempty compact subset of K. Let x_0 be an element in M, and let m be a left invariant mean on $L(S)$. We define $\mu \in C(X)^*$ by $\mu(f) = m(x_0 \otimes f)$, then $\|\mu\| = \mu(1) = 1$, $\mu \geqq 0$, and $\mu(f \circ s) = \mu(f)$ for every $s \in S$ and $f \in C(X)$ where $(f \circ s)(x) = f(sx)$.

Let F be the support of μ, that is, the unique nonempty closed subset of X such that $f \geqq 0$ vanishes on F if and only if $\mu(f) = 0$. It is easy to see that $F \subset M$. We shall show that F is S-invariant. Suppose, on the contrary, that $sx \in M - F$ for some $x \in F$ and $s \in S$. Then there exists $f \geqq 0$ such that $f(sx) > 0$ and $f(F) = 0$. Since $x \in F$ and $(f \circ s)(x) = f(sx) > 0$, we have $\mu(f \circ s) > 0$. But this

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