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QUASIHARMONIC DEGENERACY OF RIEMANNIAN N-MANIFOLDS

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The harmonic classification of Riemannian manifolds of higher dimension was recently brought to completion by the construction of manifolds which carry bounded but no Dirichlet finite nonconstant harmonic functions (Kwon [2], Hada-Sario-Wang [1]). In contrast, almost nothing is known about the quasiharmonic classification of higher dimensional manifolds. The purpose of the present study is to establish inclusion relations constituting such a classification.

Let Q be the class of quasiharmonic functions [6], i.e., solutions of $\Delta u=1$, where $\Delta=d\delta+\delta d$ is the Laplace-Beltrami operator. Denote by QP, QB, QD, and QC the classes of quasiharmonic functions which are positive, bounded, Dirichlet finite, and bounded Dirichlet finite, respectively. The notation $\tilde{O}_{QX}^N$, O_{QX}^N, with $X=P$, B, D, or C, will be used for the classes of Riemannian manifolds M of dimension $N>2$ for which $QX(M)=\emptyset$ or $\neq\emptyset$, respectively. The class of parabolic N-manifolds, characterized by the nonexistence of Green's functions, is designated by O_G^N. We shall prove that the complete classification scheme

$$O_G^N<O_{QP}^N<O_{QB}^N\cap O_{QD}^N<O_{QB}^N\,,\ O_{QD}^N<O_{QB}^N\cup O_{QD}^N=O_{QC}^N$$

holds for every N. Here "$<$" signifies strict inclusion and "O_{QB}^N, O_{QD}^N" means that the inequalities are valid for both classes.

1. We shall cover the relations in the order given above. Regarding O_G^N and O_{QP}^N, every $u\in QP$ is superharmonic by $\Delta u=1$ and its existence therefore implies that of the Green's functions (see e.g. [7]). To prove $O_G^N<O_{QP}^N$, it thus suffices to find a hyperbolic N-manifold with $QP=\emptyset$. The trivial example of the Euclidean N-space does not qualify for $N=2$, whereas the following example is valid for all N. It is an N-torus cut open along a pair of opposite faces and equipped with a suitable metric.

LEMMA 1. *Let T be the space*

$$0<x<1\,,\qquad |y_i|\leqq 1\,,\qquad i=1,\cdots,N-1\,,$$

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