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ON THE EIGHTH COEFFICIENT OF UNIVALENT FUNCTIONS, II

BY MITSURU OZAWA AND YOSHIHISA KUBOTA

0. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

For the eighth coefficient a_8 several authors had proved its local maximality at the Koebe function $z/(1-z)^2$ [1], [4], [6]. One of the authors had improved the method in [4] to a great extent in [6]. Obrock [5] and Schiffer [8] proved a general result independently, which can be formulated in the present case in the following manner: If a_2, a_3 and a_4 are real, then $\Re a_8 \leqq 8$ with equality holding only for $z/(1-z)^2$. In [7] we have given the following fact: If a_2 is real non-negative, then $\Re a_8 \leqq 8$ with equality holding only for $z/(1-z)^2$.

In this paper we shall prove the following theorem:

THEOREM. *If $a_3-3a_2^2/4$ and $a_4-3a_2a_3/2+5a_2^3/8$ are real and $|\arg a_2|\leqq\pi/7$, then $\Re a_8\leqq 8$. Equality occurs only for the Koebe function $z/(1-z)^2$.*

By the well known rotation this theorem implies the result due to Obrock and Schiffer as a simple corollary. Our original motivation in [7] and in this paper lies to investigate the status of the general a_8 problem. So the theorems are only byproducts of our original intention. We believe at the present time that the status became almost clear.

Section 1 is devoted to several preparatory lemmas and inequalities from which we start. Section 2 to 9 are concerned with the case $1\leqq\Re a_2\leqq 2$. The main part in this paper consists of sections 2, 4, 6 and 8. Section 10 is concerned with the case $0\leqq\Re a_2\leqq 1$. Sections 3, 5, 7, 9 and 10 are rather trivial parts and easy to handle.

1. We make use of the same notations as in [4]. By our assumption $y'=\eta'=0$ and $|x'/p|\leqq\tan(\pi/7)$.

First we shall give here several lemmas, which will be used later on.

LEMMA 1. $11(\tau^2+\tau'^2)+9(\varphi^2+\varphi'^2)+7(\xi^2+\xi'^2)+5\eta^2+3y^2+x'^2\leqq 4x-x^2$.

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

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