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FIXED POINTS OF REVERSIBLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS

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1. Introduction.

Takahashi [7, p. 384] proved that if K is a compact convex subset of a Banach space, and S is a left amenable semigroup of nonexpansive self-maps of K, then K contains a common fixed point of S. This theorem generalizes a result of DeMarr [2, p. 1139], who obtained the above implication for the case where S is commutative. In this note, we observe that Takahashi's theorem can be further extended, and the proof slightly simplified, by considering a purely algebraic property that every left amenable semigroup must possess, that of left reversibility. The proof employs suitable modifications of the methods of [2] and [7].

2. Fixed point theorem.

A semigroup S is called *left reversible* if for every pair of elements $a, b \in S$, there exists a pair $c, d \in S$ such that $ac=bd$. (This terminology is due to Dubreil, see [1, p. 34]).

THEOREM. *Let K be a nonempty compact convex subset of a Banach space. If S is a left reversible semigroup of nonexpansive self-maps of K, then K contains a common fixed point of S.*

Proof. A Zorn's lemma argument establishes that there exists a minimal S-invariant nonempty compact convex set $X \subseteq K$. A second application of Zorn's lemma yields that there exists a minimal S-invariant nonempty compact set $M \subseteq X$.

Since S is left reversible, a straightforward induction argument shows that if $\{s_1, s_2, \cdots, s_n\}$ is any finite subset of S, there exists a finite subset $\{t_1, t_2, \cdots, t_n\}$ of S such that $s_1t_1=s_2t_2=\cdots=s_nt_n$. Hence

$$\bigcap_{i=1}^{n} s_iM \supseteq \bigcap_{i=1}^{n} s_i(t_iM)=s_1t_1M \neq \phi.$$

Thus the family $\{sM;\ s\in S\}$ has the finite intersection property, so defining $F=\bigcap\{sM;\ s\in S\}$, we get that F is nonempty by compactness of M.

Let $x\in F$. For each pair $a, b\in S$, there exists $c, d\in S$ such that $ac=bd$. Since $F\subseteq cM$, then $x=cy$ for some $y\in M$. Hence

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