ON BIRECURRENT TENSORS

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It has been recently proved[1] that if, in a compact Riemannian manifold of class C^∞, a tensor field T satisfies the covariant differential equation $\nabla_j \nabla_i T = \beta_j \nabla_i T$, then the covariant derivative of the tensor T vanishes identically. On the other hand, it is known [2][2] that if, in a complete irreducible Riemannian manifold of class C^∞, the m-th covariant derivative of any tensor field for some $m \geqq 1$ vanishes identically, then so does the covariant derivative of the tensor field.

In connection with these results, Prof. Yano suggests to study the tensor field T satisfying the covariant differential equation

$$\nabla_m \nabla_l T + \alpha_{ml} T = 0, \tag{1}$$

α_{ml} being a tensor field of type (0, 2). The main purpose of this note is to get a sufficient condition for such a tensor field to vanish identically.

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§1. Let M be an n-dimensional Riemannian manifold of class C^∞ covered by a system of local coordinates $\{x^h\}$[3] and g_{ji} the components of the metric tensor. Let all vector fields and tensor fields be of class C^∞. If a tensor T, say T_{kji}, satisfies the equation (1), we call T a recurrent tensor of second order (or briefly a birecurrent tensor) and α_{ml} the associated tensor of the birecurrent tensor T.

For any birecurrent tensor the associated tensor α_{ml} is symmetric. In fact, transvecting T^{kji} to $\nabla_m \nabla_l T_{kji} + \alpha_{ml} T_{kji} = 0$, we get

$$\nabla_m \nabla_l (T_{kji} T^{kji})/2 - \nabla_m T^{kji} \cdot \nabla_l T_{kji} + \alpha_{ml} T_{kji} T^{kji} = 0.$$

Since the first and the second terms in the first member are symmetric in m and l, so is the last term, that is, α_{ml} too.

First we consider the case in which the tensor field S, say S_{kji}, satisfies covariant differential equations of slightly general form

$$\nabla_m \nabla_l S_{kji} + \beta_m \nabla_l S_{kji} + \alpha_{ml} S_{kji} = 0, \tag{2}$$

β_m being a vector field. Transvecting $g^{ml} S^{kji}$ to (2) and putting $S^2 = S_{kji} S^{kji}$, we get

$$\Delta S^2 - 2\nabla_m S_{kji} \cdot \nabla^m S^{kji} + \beta^m \nabla_m S^2 + 2\alpha_m{}^m S^2 = 0,$$

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1) Nomizu and Yano, personal communication.

2) Numbers in brackets refer to the bibliography at the end of the note.

3) Throughout this note, indices run over the range 1, 2, $\cdots$, n.