CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA, III

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1. Introduction.

The theory of rings of operators of von Neumann has been developed by many authors, especially since it has been regarded as a non-commutative extension of the integration over a measure space by Dixmier [3],[1] Dye [4] and Segal [10], some fundamental theorems on measure theory have been extended. In the papers [3], [8] and [14], the authors have introduced the concept of the conditional expectation into some ring of operators of von Neumann (=von Neumann algebra in the sense of Dixmier [2] and we shall use this terminology below) which can be also regarded as a non-commutative extension of conditional expectation in the probability theory. The extension has also been made of the C. Moy characterization theorem (cf. [8] and [14]) and the martingale convergence theorem (cf. [15]). In the present note, as a part of a non-commutative extension of measure theoretic probability theory we shall prove for a von Neumann algebra of finite class a Halmos-Savage theorem (cf. [5]) with respect to sufficient statistics in probability theory which was reformulated under the terminology of Borel subfield by Bahadur [1].

For our purpose we shall depend upon as a basic theorem the Radon-Nikodym theorem due to Dye [4], in a von Neumann algebra. Firstly we shall introduce a space of some restricted normal states relative to a von Neumann subalgebra which will be called tracelet space (cf. Definition 1) and give an example of such space by the direct product of finite factors in the sense of Nakamura [7], and further we shall extend the existence theorem of the conditional expectation in a von Neumann algebra for normal states in the tracelet space (cf. Theorem 1). Under Theorem 1, we shall introduce a concept of sufficiency of von Neumann subalgebra for some restricted set of normal states and prove a Halmos-Savage characterizations of sufficient subalgebra (cf. Theorems 5 and 6). These have applications to find a characteristic property of subalgebra having unique expectation onto it (cf. Theorem 7), and to prove a von Neumann proposition (cf. Theorem 2 of [9]) for a von Neumann's operations which are stated in the final section.

2. Preliminary.

Let A be a countably decomposable ($=\sigma$-finite) von Neumann algebra of

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1) The numbers in brakects refer to the reference at the end of this paper.