SOME LIMIT THEOREMS CONCERNING WITH THE RENEWAL NUMBERS

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Let $\{X_i\}$ be a sequence of independent random variables and let $\{a_i\}$ be a sequence of real numbers. Denoting as $S_n=\sum_1^n X_i$, we shall interest the weighted mean of renewal numbers in the interval $(a, x+h)$, which is defined by

$$A(x, h)=\sum_{n=1}^{\infty} a_n P(x< S_n \leqq x+h). \tag{1}$$

If both $\{E(X_i)\}$ and $\{a_i\}$ are stable sequences[1] with average m and a, respectively, and if some further conditions are satisfied, then it is known that $A(x, h)\to ah/m$ $(x\to\infty)$ by Cox and Smith [1].

But when $\{E(X_i)\}$ is not stable, $A(x, h)$ is not necessarily convergent to a finite limit as $x\to\infty$. In this case, instead of $A(x, h)$, the variable

$$\bar{A}_h(X)=\frac{1}{X}\int_0^X A(x, h)\,dx$$

will converge to ah/m as $X\to\infty$, under suitable conditions. This fact was shown by the analogous argument of [2] by Prof. T. Kawata.

From a practical problem it was necessary to us to find the distribution of $A(x, h)$ or $\bar{A}_h(X)$ when $\{a_i\}$ is a sequence of independent random variables having the mean a. We shall treat in the present paper this problem when a_i are the random variables identically distributed and obeying the exponential distribution.

First of all, we shall prepare the following lemmas.

LEMMA 1. *Let X_i $(i=1, 2, \cdots)$ be independent random variables having the distribution function $F_i(x)$ such that $E(X_i)=m_i>0$. Suppose that the following conditions are satisfied:*

$$\int_{-\infty}^{0} e^{-sx}dF_i(x)<\infty \qquad \textit{for} \quad 0\leqq s\leqq s_0, \tag{2}$$

$$\lim_{A\to\infty}\int_A^{\infty} x\,dF_i(x)=0, \tag{3}$$

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1) Cox and Smith [1] gave the following

DEFINITION. A sequence $\{\mu_i\}$ such that $\lim_{p\to\infty}\frac{1}{p+1}\sum_{i=n}^{n+p}\mu_i=\mu$, uniformly in n, will be called *stable* with average μ.