ON FUNCTIONS OF BOUNDED DIRICHLET INTEGRAL

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Ozawa [1] has derived a perfect criterion in terms of local coefficients in order that a single-valued regular function has an image domain whose area does not exceed π. In the present paper we shall notice that the criterion may be obtained as a particular case of a topological theorem due to Helly [2]. We shall further show that from this point of view it can be extended functions defined on a Riemann surface.

1. Basic notations. Let B be a planar n-ply connected schlicht domain with a boundary Γ consisting of analytic curves Γ_ν ($\nu = 1, \ldots, n$). For the sake of simplicity, suppose that B contains the origin.

Let z_0 be any assigned point in B and $P(z, z_0)$ be a polynomial with respect to $t \equiv 1/(z - z_0)$:

$$P(z, z_0) = \sum_{m=1}^{N} x_m t^m.$$

Let further α be a real parameter and $f_P(z, z_0; \alpha)$ be a single-valued meromorphic function characterized by the following conditions:

i. $f_P(z, z_0, \alpha) - P(z, z_0)$ is regular in B and vanishes at z_0;
ii. all the images of Γ_ν ($\nu = 1, \ldots, n$) by $f_P(z, z_0, \alpha)$ are segments with inclination α to the real axis.

Existence of $f_P(z, z_0, \alpha)$ for any given P and α, together with its uniqueness, is well-known; cf. Grunsky [3].

Put

$$F_P(z, z_0; \alpha) = \frac{1}{2}\left(f_P(z, z_0, \alpha) - f_P(z, z_0, \alpha + \frac{\pi}{2})\right),$$

then we have

$$F_P(z, z_0, \alpha) = e^{2i\alpha} \sum_{m=1}^{N} \bar{x}_m \varphi_m(z, z_0),$$

where

$$\varphi_m(z, z_0) = \frac{1}{2}\left(f_{t^m}(z, z_0, 0) - f_{t^m}(z, z_0, \frac{\pi}{2})\right).$$

Put further

$$\Phi_m(z, z_0) = \frac{1}{2}\left(f_{t^m}(z, z_0, 0) + f_{t^m}(z, z_0, \frac{\pi}{2})\right).$$

The local expansions of Φ_m and φ_m ($m \geq 1$) about z_0 are obviously of the forms

$$\Phi_m(z, z_0) = \frac{1}{(z - z_0)^m} + \sum_{\nu=1}^{\infty} B_{m\nu} (z - z_0)^\nu,$$

and

$$\varphi_m(z, z_0) = \sum_{\nu=1}^{\infty} S_{m\nu} (z - z_0)^\nu,$$

respectively. There holds

$$d\varphi_m = d\overline{\Phi_m} \qquad \text{along } \Gamma.$$

Let $L^2(B)$ be a family of functions $\psi(z)$ satisfying the following conditions:

i. $\Psi(z) \equiv \int^z \psi(z)\, dz$ is a single-valued function regular in B;
ii. $\iint_B |\psi(z)|^2 d\sigma_z < \infty$, $d\sigma_z$ denoting the areal element.

In $L^2(B)$ we define the Dirichlet norm by

$$\|\psi\|_B = D_B(\Psi, \Psi)^{\frac{1}{2}} \equiv \left(\iint_B |\psi(z)|^2 d\sigma_z\right)^{1/2},$$

and denote by $D_B(\Psi_1, \Psi_2)$ the associated bilinear integral form:

$$D_B(\Psi_1, \Psi_2) = \iint_B \Psi_1'(z) \overline{\Psi_2'(z)}\, d\sigma_z.$$

Introducing further the inner product by

$$(\psi_1, \psi_2) = D_B(\Psi_1, \Psi_2),$$

$$\Psi_j(z) \equiv \int^z \psi_j(z)\, dz \quad (j = 1, 2),$$

$L^2(B)$ becomes a Hilbert space.