IDENTITIES CONCERNING CANONICAL CONFORMAL MAPPINGS

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1. In general, a domain of finite connectivity which is finitely many-sheeted and possesses no degenerate boundary component is called a Riemann half-surface, if the projection of its whole boundary lies on a fixed circle alone.

Every Riemann half-surface becomes a closed Riemann surface, when it is inverted with respect to any one of its boundary components and the original boundary is then sewn with its inverted image by idential coordination. We may suppose further without loss of generality that the fixed circle bearing the projection of the boundary of a Riemann half-surface is coincident with the real axis of a complex plane. The closed Riemann surface obtained by the duplication process is then generated by an irreducible algebraic equation with merely real coefficients.

Let now $\mathfrak{h}$ be a Riemann half-surface laid over the w-plane, and $\mathfrak{F}$ be the closed Riemann surface obtained from $\mathfrak{h}$ by the duplication process. Let further

$$E(w, \mathfrak{w}) = 0$$

be the irreducible algebraic equation generating $\mathfrak{F}$ and

$$\mathfrak{w} = A(w)$$

be the algebraic function defined by the equation. Every analytic function meromorphic on the whole surface $\mathfrak{F}$ is necessarily a rational function with respect to w and $\mathfrak{w}$.

Let $\mathfrak{h}^*$ be any Riemann half-surface laid over the w^*-plane, yielding the corresponding closed Riemann surface $\mathfrak{F}^*$. If $\mathfrak{h}^*$ is conformally equivalent to $\mathfrak{h}$, then any analytic function

$$w^* = f(w)$$

mapping $\mathfrak{h}$ onto $\mathfrak{h}^*$ is surely prolongable, in virtue of inversion principle, beyond its boundary into $\mathfrak{F}$, and the function thus prolonged maps, of course, the whole surface $\mathfrak{F}$ onto $\mathfrak{F}^*$. Hence, according to the fact mentioned above, it must be a rational function with respect to w and $\mathfrak{w}$, i.e. there holds a relation of the form

$$f(w) = R(w, A(w)),$$

where $R(w, \mathfrak{w})$ designates a rational function with respect to its both arguments.

2. Let now D be any domain, laid on the z-plane, with no degenerate conponent boundary. We consider two analytic functions

$$w = w(z) \quad \text{and} \quad w^* = w^*(z)$$

which map D onto the Riemann half-surfaces $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively. Consequently, $\mathfrak{h}$ and $\mathfrak{h}^*$ are conformally equivalent. The function $w^* = f(w)$ obtained by eliminating z form both relations $w = w(z)$ and $w^* = w^*(z)$ maps rally $\mathfrak{h}$ onto $\mathfrak{h}^*$, and hance it is of the nature mentioned above. We thus conclude that a relation of the form

$$w^*(z) = R(w(z), A(w(z)))$$

must hold, R designating, as stated above, a rational function of its both arguments. The last relation can be regarded as a functional dependence between the mapping functions $w(z)$ and $w^*(z)$.

Analytic functions mapping a given basic domain D onto Riemann half-surfaces can be constructed in various ways, especially in connection with the functions mapping D onto canonical domains of several types. Such a method has indeed been availed already by H. Lenz[1] for proving the Schottky's mapping theorem. Following his idea and applying it to more gener general classes, we shall illustrate in the present Note several examples of such functions, among which the functional relations of the above-mentioned form must be valid.

3. Before entering into the main discourse, we begin with giving an immediate example. Suppose that D is an n-ply connected jordan domain bounded by n disjoint contours C_j