ON PRIMAL ELEMENTS IN A MODULAR LATTICE

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The concept of primal ideals, introduced by L. Fuchs (3) for commutative rings and generalized by C. W. Curtis (2) to integral modular lattice ordered semigroups with ascending chain condition, shall be extended in this paper to modular lattices with maximum condition by our method in (4).

§ 1.

Let L be a modular lattice with ascending chain condition and Θ be a set of lattice congruences on L such that any meet of a finite number of congruences in Θ is also in Θ. We denote by $a(\theta)$ the greatest element congruent to an element a by a congruence θ on L. The set of elements x satisfying $x(\theta) = 1$ is denoted by $\chi(\theta)$. An element q is said to be primary (with respect to Θ) if $q \in \chi(\theta)$ or $q = q(\theta)$ for every θ in Θ. An element a is called to be primal (with respect to Θ) if $a(\theta_1 \cap \theta_2) = a$ implies $a(\theta_1) = a$ or $a(\theta_2) = a$ for θ_1 and θ_2 in Θ.

Theorem 1.1. Any primary element is primal.

Proof. Let q be a primary element. $q(\theta_1) > q$ and $q(\theta_2) > q$ imply $q(\theta_1) = 1$, $q(\theta_2) = 1$ and $1 > q$, that is, $q(\theta_1 \cap \theta_2) = 1 > q$.

Theorem 1.2. Any meet-irreducible element is primal.

Proof. If an element a is not primal then $a(\theta_1 \cap \theta_2) = a$, $a(\theta_1) > a$ and $a(\theta_2) > a$ for some θ_1, θ_2 in Θ. Since $a(\theta_1) \cap a(\theta_2) = a(\theta_1 \cap \theta_2) = a$, a is meet-reducible.

In a meet of elements of L, if we can not replace any component by a element greater than it, we call the meet to be reduced. An irredundant meet of primal elements is said to be shortest if any meet of two or more components is not primal. Reduced and shortest meets are normal.

Theorem 1.3. Let $a = a_1 \cap a_2 \cap \dots \cap a_n$ be a reduced meet. Then $a(\theta) = a$ if and only if $a_i(\theta) = a_i$ for $i = 1,2,\dots,n$.

Proof. If $a_i(\theta) = a_i$ for $i = 1,2,\dots,n$ then $a(\theta) \leq a_i$ and hence $a(\theta) = a$. Conversely, suppose $a(\theta) = a$. Now $a_1(\theta) \cap a_2(\theta) \cap \dots \cap a_n(\theta) \geq a(\theta)$ and the left hand side is congruent to a by θ. Hence $a = a_1(\theta) \cap a_2(\theta) \cap \dots \cap a_n(\theta)$. From the reducibility, $a_i(\theta) = a_i$ for $i = 1,2,\dots,n$.

Theorem 1.4. Let $a = a_1 \cap a_2 \cap \dots \cap a_n$ be a reduced meet of primal elements a_i. Then a is primal if and only if $a_i(\theta) = a$ implies $a(\theta) = a$ for any θ in Θ and some integer i independent to θ.

Proof. By the theorem 1.3. $a(\theta) = a$ ensures that $a_i(\theta) = a$ for $i = 1,2,\dots,n$. If the condition is satisfied then a is primal by the definition. If it is not satisfied then there are θ_i such that $a_i(\theta_i) = a_i$ and $a(\theta_i) > a$ for $i = 1,2,\dots,n$. $a \leq a(\theta_1 \cap \theta_2 \cap \dots \cap \theta_n) \leq a(\theta_i) \leq a_i(\theta_i) = a_i$. Hence $a(\theta_1 \cap \theta_2 \cap \dots \cap \theta_n) = a$. If a were primal then $a(\theta_i) = a$ for some i which is a contradiction.

Theorem 1.5. Any element is expressible as a normal meet of a finite number of primal elements.

Proof. First, if we represent the element as an irredundant meet of meet-irreducible elements, it is necessarily reduced. Next, by the Theorem 1.4., grouping its suitable components we obtain a shortest meet. It is easy to see the reducibility of this meet. (Only here we use the modularity of L.)

For any element a, the set of θ in Θ satisfying $a(\theta) = a$ is a M-closed subset of Θ which denoted by M(a). It is well known that the set of all M-closed subsets in Θ forms a distributive lattice, by set-inclusion, to which we refer as $M(\Theta)$. The definition shows that an element a is primal if and only if M(a) is meet-irreducible in $M(\Theta)$. If $a = a_1 \cap a_2 \cap \dots \cap a_n$ is a normal meet of primal a_i, then, by the theorem