A GENERALIZATION OF ABSOLUTE NEIGHBORHOOD RETRACTS

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1. In this paper we give some generalization of absolute neighborhood retracts [1)] [1]. This generalization is not useful on homotopy theory but admits some generalizations on fixed point properties of ANRsets (section 4). In sections 2 and 3 the familiar definitions and theorems of ARsets and ANRsets are described with the slight modifications.

2. In the following a space and a set are always separable metric.

(2.1) DEFINITION. Given the number ε', $\varepsilon' > 0$, and the sets A and B such that $B \subset A$ we say that a map $r_{\varepsilon'}$ is an ε'-retraction provided $r_{\varepsilon'}$ is defined and continuous on A, $r_{\varepsilon'}(B) \subset B$ and $r_{\varepsilon'}(A) \subset B$, and $\rho(b, r_{\varepsilon'}(b)) < \varepsilon'$ [2)] for every $b \in B$. If such maps exist for every $\varepsilon' > 0$, then B is called an ε-retract of A.

(2.2) DEFINITION. Given the sets A and B such that $B \subset A$, we say that B is an ε-neighborhood retract of A provided there exists an open set U such that $B \subset U \subset A$ and such that B is an ε-retract of U.

(2.3) DEFINITION. A space, A, is called an ε-absolute neighborhood retract (ε-ANR or ε-ANRset) provided it is a compactum and for every topological image A_1 of A, such that A_1 is contained in a space M, we have A_1 is an ε-neighborhood retract of M.

(2.4) THEOREM. <u>A necessary and sufficient condition for a set to be an ε-ANR is that it be homeomorphic to a closed ε-neighborhood retract of the Hilbert parallelotope Q.</u>

PROOF. Necessity. Let A be an ε-ANR. Since A is a compactum, we can map A topologically into the Hilbert parallelotope Q [5]. Let $h(A) = A_1$, where h is a homeomorphism and A_1 is a subset of Q. Since Q is a compactum, by (2.3) A_1 is an ε-neighborhood retract of Q. In virtue of the continuity of h and the compactness of A, we have A_1 is compact and therefore closed in Q.

Sufficiency. Let $h(A) = A_1$, where h is a homeomorphism and A_1 is a closed ε-neighborhood retract of Q. Consider any other homeomorphic image A_2 of A such that A_2 is contained in a space M. Let $k(A) = A_2$, where k is a homeomorphism. Q is a compactum. Therefore A_2 is a compactum and hence closed in M. We now apply Tietze's extension theorem [5] to the map $hk^{-1} : A_2 \longrightarrow Q$ and obtain an extension f of hk^{-1} over M relative to Q. Since A_1 is an ε-neighborhood retract of Q, there exists an open set $U_1 \supset A_1$ and for each $\varepsilon' > 0$ an ε'-retraction $r_{\varepsilon'}$ such that $r_{\varepsilon'} : U_1 \longrightarrow A_1$. Now $f^{-1}[f(M) \cap U_1]$ is an open subset of M and clearly $f^{-1}[f(M) \cap U_1] \supset A_2$. The map $kh^{-1} r_{\varepsilon'} f$ maps the open set $f^{-1}[f(M) \cap U_1]$ into A_2. Since A_2 is compact, for sufficiently small ε' by uniform continuity of $kh^{-1} r_{\varepsilon'} f$ we have

$$\rho(a, kh^{-1} r_{\varepsilon'} f(a)) < \varepsilon \quad \text{for every } a \in A_2,$$

where ε is any giving positive number. Thus A_2 is an ε-neighborhood retract of M.

(2.5) DEFINITION. A space, A, is called an ε-absolute retract (ε-AR or ε-ARset) provided it is a compactum and for every topological image A_1 of A, such that A_1 is contained in a space M, we have A_1 is an ε-retract of M.

(2.6) THEOREM. <u>A necessary and sufficient condition for A to be an ε-AR is that it be homeomorphic to a closed ε-retract of the Hilbert parallelotope Q.</u>

This result may be verified by the method of (2.4).

(2.7) In (2.4) and (2.6) when the dimension of A is finite we can replace Q by a sufficiently high dimensional Euclidean space. Naturally every ANR(AR) set is an ε-ANR(ε-AR) set.

(2.8) EXAMPLE. In two-dimensional Euclidean space we consider next set A in a rectangle x-y coordinate.