STRUCTURE OF HOMOGENEOUS CHAINS

By Tadashi OHKUMA

In the previous paper [3], the author determined the structure of discrete homogeneous chains. Continuing his study, he will show the structure of general homogeneous chains to some extent, in the present paper.

In the structure theory of homogeneous chains, simple homogeneous chains, which will be defined later, seem to be fundamental, and some attempt of the representation of general homogeneous chains, as lexicographic product of simple homogeneous chains is suggested at the end of this paper.

As to definitions and notations, the same as used in [1] and in the author's previous paper, [3] are employed. But, for convenience, a short note of these definitions and the results gained from these definitions, which are used in the author's previous paper, are stated again in §1.

In the study of the structure of homogeneous chains, the homogeneous intervals play an important rôle, and §1 is devoted to the investigation of the homogeneous intervals in a homogeneous chain.

In §2, homogeneous chains, which have very special type, namely, homogeneous chains with unique automorphisms, are studied.

In §3 the structure of simple homogeneous chains, especially that of conditionally complete homogeneous chains, which belongs to this category, are determined to some extent.

Those are homogeneous chains with very special type, but a general homogeneous chain is embedded in a lexicographic product of these simple homogeneous chains. The fact is shown in the last section, §4, of this paper.

§1. Homogeneous intervals.

The terms used without definitions, such as a partially ordered set, (abbr. a poset), a chain (or a totally ordered set), and an ordinal number (or a well-ordered set), ought to be referred to [1].

(1.1) Definition 1. If a chain X has a transitive automorphism group, we call X homogeneous.

Definition 2. A subchain I of a chain X is called an interval of X. if and only if,

$$a,\ b \in I \quad \text{and} \quad a < c < b$$

$$\text{implies} \quad c \in I.$$

The whole chain X and a subchain which consists of only one element of X are intervals. The other intervals are called proper.

Especially, for any pair of elements a, b of X, the set of elements between (properly) a and b is an interval of X, which we call an open interval (a, b). The set of upper bounds and the set of lower bounds of an element a of X, excluding the element a, are also called (unbounded) open intervals, and are denoted by $(a,-)$ and $(-, a)$ respectively. When two elements a and b are adjoined to the open interval (a, b), we call it a closed interval $[a, b]$. $[a, b)$ denotes the interval (a, b) with adjoined a only. $(a, b]$, $[a, -)$, and $(-, a]$ are similarly defined.

(1.2) We define the following two kinds of orders in a family of intervals of a chain X.

P.1) We say that Y_1 contains Y_2, if and only if Y_2 is a subset of Y_1, and denote the fact by $Y_2 \subset Y_1$. (Or, we may say that Y_1 is greater than Y_2 in the meaning of P.1).)

P.2) We say that Y_2 is less than Y_1 (or Y_1 is greater than Y_2) if and only if $a < b$ for any pair of $a \in Y_2$, and $b \in Y_1$, and denote the fact by $Y_2 < Y_1$. (Or, precisely, we say that Y_1 is greater than Y_2, in the meaning of P.2).) Especially the subset of X, which consists of only one element $x \in X$ is an interval of X. If x is less than any element of the other interval Y of X, the fact is denoted by $x < Y$. The sign $x > Y$ is similarly defined.

Y_1 and Y_2 are comparable if and only if either they are disjoint or they coincide entirely with each other.