ON PRIMARY ELEMENTS OF A MODULAR LATTICE

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It is well known that, in a Noetherean ring, every ideal can be written as an intersection of primary ideals. The theorem was extended by M.Ward and R.P.Dilworth to integral modular lattice ordered semigroups with maximum condition.([2], [3]) The purpose of this paper is to discuss it for modular lattices with maximum condition.

§ I. Definitions.

Let L be a modular lattice with maximum condition and Θ be a set of congruences on L such that every meet of congruences in Θ is also in Θ. For a congruence θ the class containing an element a is denoted by $\theta(a)$ and the greatest element of $\theta(a)$ by a_θ.

Definition I. An element q of L is primary (with respect to Θ) if and only if $q = q_\theta$ or $q \equiv I(\theta)$ for every θ in Θ.

Definition 2. A congruence θ in Θ is a radical (with respect to Θ) of an element a in L if and only if θ is the smallest one among the congruences by which a is congruent to I.

The radical of a is denoted by $\rho(a)$. Evidently, $\theta \geq \rho(a)$ if and only if $a \equiv I(\theta)$. For a primary element q we have $q \equiv I(\theta)$ if $\theta \geq \rho(q)$, and $q_\theta = q$ if $\theta \ngeq \rho(q)$.

Definition 3. By a short representation (with respect to Θ) of an element a in L, it is meant a representation of a as an irredundant meet of a finite number of primary elements all of which radicals are different.

Definition 4. A congruence on L is said to be neutral if (a) the class containing I is a neutral dual ideal ([I]) and (b) a is congruent to b if and only if $a \wedge x = b \wedge y$ for some x and y congruent to I.

§ 2. UNIQUENESS

Lemma I. $(a \wedge b)_\theta = a_\theta \wedge b_\theta$

Theorem I. Let $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ where every q_i be primary and θ be in Θ. If $q_j \not\equiv I(\theta)$ for $j = 1, 2, \cdots, r$ and $q_k \equiv I(\theta)$ for $k = r+1, \cdots, n$ then $a_\theta = q_1 \wedge q_2 \wedge \cdots \wedge q_r$

Proof. By the assumption, $q_{r+1} \wedge \cdots \wedge q_n \equiv I(\theta)$ and hence $a = q_1 \wedge q_2 \wedge \cdots \wedge q_r (\theta)$. But $q_{j\theta} = q_j$ for $j = 1, 2, \cdots, r$. Thus $a_\theta = q_{1\theta} \wedge q_{2\theta} \wedge \cdots \wedge q_{r\theta} = q_1 \wedge q_2 \wedge \cdots \wedge q_r$

Theorem 2. In two short representations of an element a in L the radicals of the components coincide.

Proof. Assume, two short representations $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n = q'_1 \wedge q'_2 \wedge \cdots \wedge q'_m$ are given. Let the minimal one among the $\rho(q_i)$ and $\rho(q'_j)$ be, say, $\rho(q_1) = \theta$. Now $a_\theta = q_{1\theta} \wedge q_{2\theta} \wedge \cdots \wedge q_{n\theta} = q'_{1\theta} \wedge q'_{2\theta} \wedge \cdots q'_{m\theta}$. If every $\rho(q'_j)$ were different from θ then $q_2 \wedge \cdots \wedge q_n = q'_1 \wedge q'_2 \wedge \cdots \wedge q'_m = a$ which contradicts our assumption. Hence among the $\rho(q'_j)$ some one, say, $\rho(q'_1)$ is equal to θ. Whence $q_2 \wedge \cdots \wedge q_n = q'_2 \wedge \cdots \wedge q'_m$ which completes the proof by the finite induction.

§ 3. DECOMPOSABILITY

Theorem 3. A meet of a finite number of primary elements which have the same radicals is also primary and has the same radical.

Proof. Assume, $q_1, q_2, \cdots, q_n$ are primary and $\rho(q_1) = \rho(q_2) = \cdots = \rho(q_n)$ Let θ be in Θ. If $\theta \geq \rho(q_1)$ then $q_i \equiv I(\theta)$ for $i = 1, 2, \cdots, n$. Hence $q_1 \wedge q_2 \wedge \cdots \wedge q_n \equiv I(\theta)$. But, if $\theta \ngeq \rho(q_1)$ then $q_i = q_{i\theta}$ for $i = 1, 2, \cdots, n$. Thus $(q_1 \wedge q_2 \wedge \cdots \wedge q_n)_\theta = q_1 \wedge q_2 \wedge \cdots \wedge q_n$. Whence $q_1 \wedge q_2 \wedge q_3 \cdots \wedge q_n$ is primary. Next, $q_1 \wedge q_2 \wedge \cdots \wedge q_n \equiv I(\rho(q_1))$ since $q_i \equiv I(\rho(q_1))$. But, if $q_1 \wedge q_2 \wedge \cdots \wedge q_n \equiv I(\varphi)$ for some φ in Θ then a fortiori $q_1 \equiv I(\varphi)$ thus $\rho(q_1) \leq \varphi$. Hence $q_1 \wedge q_2 \wedge \cdots \wedge q_n$ has the radical $\rho(q_1)$.

Theorem 4. An irredundant meet of a finite number of primary elements of which not all have the same radicals is not primary.

Proof. We may assume by th. 3 that all the radicals of primary components are different. Let $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ be irredundant where q_i be primary. If the minimal one among $\rho(q_i)$ is, say, $\rho(q_1) = \theta$ then