ON SOME FAMILY OF MULTIVALENT FUNCTIONS

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1. Quasi - convex Functions.

Let

$$(1) \quad F(z) = z^p + \sum_{K=1}^{\infty} a_{p+K} z^{p+K}$$

be any function regular in $|z| < 1$, where p is a positive integer.[1] If we denote a family of functions of the form (1), by which $|z| < 1$ is transformed into a starshaped (with the center at the origin) or a convex region of p-valence, respectively denoted by $\mathfrak{S}_p$ or $\mathfrak{K}_p$, then the following theorem is well known.[2]

Theorem 1.

The necessary and sufficient condition that $F(z)$ should belong to $\mathfrak{S}_p$ or $\mathfrak{K}_p$ is that

$$R\left[z\frac{F'(z)}{F(z)}\right] > 0$$

or

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] - \frac{p-1}{p} R\left[z\frac{F'(z)}{F(z)}\right] > 0$$

holds respectively in $|z| < 1$.

Now we denote by $\mathfrak{Q}_p$ a family of functions of the form (1) which is characterized by the following properties:

1° The mapped region of $|z| < 1$ by $w = F(z)$ is p-valent,

2° The curvature at any point on the mapped curve of $|z| = r$ by $w = F(z)$ is positive and finite, where r is an arbitrary positive number less than unity.

And we say that $F(z)$ in $\mathfrak{Q}_p$ is a quasi-convex function, then we have the theorem as follows:

Theorem 2.

The necessary and sufficient condition that $F(z)$ should belong to $\mathfrak{Q}_p$ is

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0 \qquad (|z| < 1).$$

Proof. We have, by (1),

$$\left[\frac{F'(z)}{z^{p-1}}\right]_{z=0} = p \neq 0.$$

Therefore, if

$$(2) \quad R\left[z\frac{(zF'(z))'}{zF'(z)}\right] = 1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0,$$

then $\frac{F'(z)}{z^{p-1}} \neq 0$ in $|z| < 1$ and $F'(z) \neq 0$ in $0 < |z| < 1$.[3] Denoting by ρ the curvature at any point given in 2°, we have

$$\rho = \frac{1}{|zF'(z)|} \cdot R\left[1 + z\frac{F''(z)}{F'(z)}\right] > 0.$$

The mapped curve C of $|z| = r$ by $w = F(z)$ is regular and the angle φ between the real axis and the tangent to the curve C at any point on C is given by $\arg izF'(z)$. Hence we have, as z describe $|z| = r$ in the positive direction,

$$\int d\arg izF'(z) = \int d\arg z^p + \int d\arg \frac{F'(z)}{z^{p-1}}$$
$$= \int d\arg z^p = 2p\pi,$$

and consequently the curve C is closed and p-valent. Here r being arbitrary, the mapped region of $|z| < 1$ is p-valent.

Conversely, if $\rho > 0$, then $1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0$ follows directly from the equality for ρ cited above. Our theorem is thus proved.

2. Relations among $\mathfrak{S}_p$, $\mathfrak{K}_p$ and $\mathfrak{Q}_p$.

Let $F(z)$ be any function regular and p-valent in $|z| < 1$,