A REMARK ON THE CLASS O_{HD} OF RIEMANN SURFACES

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Let F be an open Riemann surface. A "subsurface" G of F means a connected open set G on F whose boundary set C consists of a finite or infinite number of compact or non-compact simple continuous curves clustering nowhere on F. Further it is assumed that the closure G + C of G has the same boundary set C as G. C is the "relative boundary" of G.

Let HB, HD and HBD denote the classes of single-valued harmonic functions in a region, which are respectively bounded, have a finite Dirichlet integral or have both of these properties.

A surface F is said to belong to the class O_{HD} if every function $u(p) \in HD$ on F reduces to a constant ([1]). The classes O_{HB} and O_{HBD} are defined similarly. It is known that $O_{HB} \subset O_{HD} = O_{HBD}$ ([6], [7]).

By analogy we shall denote by SO_{HD} the class of subsurfaces G with the relative boundary C, such that every function u(p) continuous on G + C, $= 0$ on C and $\in$ HD in G vanishes identically. Two subsurfaces G and G^* (of two surfaces F and F^* respectively) are identified when there exists a one-to-one and conformal transformation between G and G^* which is one-to-one and bicontinuous also on the closures of G and G^*. The classes SO_{HB} and SO_{HBD} are defined similarly, and again there holds $SO_{HB} \subset SO_{HD} = SO_{HBD}$ ([4]).

Let $\{F_n\}_{n=0,1,\dots}$ be an exhaustion of F, and Γ_n be the boundary of F_n. Let $\omega_n(p, G)$ denote the harmonic measure of $G\cdot\Gamma_n$ with respect to $G\cdot F_n$, and put $\omega(p, G) = \lim_{n\to\infty} \omega_n(p, G)$. It is immediately proved that $G \in SO_{HB}$ and $\omega(p, G) \equiv 0$ are equivalent. Further, if $\omega(p, G) \not\equiv 0$ then $\sup \omega(p, G) = 1$. (Cf. [4], [5] and [6].)

Let G, G' be a pair of disjoint subsurfaces of F. It is known that, if both G and G' are not of class SO_{HD}, then $F \notin O_{HD}$ ([3], [4]). In this note we shall prove:

Theorem. If $G \notin SO_{HD}$ and $G' \notin SO_{HB}$, then $F \notin O_{HD}$.

Remark. Whether the inclusion $O_{HB} \subset O_{HD}$ be proper or not remains still unknown. As for the classes of subsurfaces, however, the following example shows that SO_{HB} is a proper subclass of SO_{HD}.

Let E be a closed set of points on the real axis of the complex z-plane, F be its complement and G be the upper half-plane $\Im z > 0$. The relative boundary C of G is the real axis deleted in E. Suppose that $G \notin SO_{HB}$ and $u(z) \in$ HB in G, $= 0$ on C and $\not\equiv 0$. By the principle of reflection u(z) and its conjugate function v(z) can be harmonically continued across C to functions defined and single-valued in F, so that $f(z) = \exp.(u(z) + iv(z))$ is a non-constant bounded analytic function in F. Conversely, if f(z) is non-constant, bounded and analytic in F, decompose f(z) into

$$f(z) = f_1(z) + if_2(z),$$

$$f_1(z) = (f(z) + \overline{f(\bar{z})})/2,$$

$$f_2(z) = (f(z) - \overline{f(\bar{z})})/2i.$$

Then, $\Im f_1(z)$ and $\Im f_2(z)$ are of class HB in G and $= 0$ on C, and at least one of these two is $\not\equiv 0$, so that $G \notin SO_{HB}$. A similar reasoning holds also for the class HD, and we have: $G \in SO_{HB}$ or $\in SO_{HD}$ if and only if $E \in N_{\mathfrak{B}}$ or $\in N_{\mathfrak{D}}$ respectively in Ahlfors-Beurling's [2] sense (i.e. $F \in O_{AB}$ or $\in O_{AD}$ in the sense of [1]). On the other hand, it is shown in [2] that there exist linear point-sets belonging to $N_{\mathfrak{D}}$ but not to $N_{\mathfrak{B}}$. Hence, the inclusion $SO_{HB} \subset SO_{HD}$ is proper.