ON NUMBERS OF POSITIVE SUMS OF INDEPENDENT RANDOM VARIABLES

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The following theorem on numbers of positive sums of independent random variables has been proved by P.Erdos and M.Kac.[1]

Theorem A. Let $X_1, X_2, \ldots$ be independent random variables each having mean 0 and variance 1 and such that the central limit theorem is applicable. Put $S_n = X_1 + X_2 + \cdots + X_k$ and let N_n denote the numbers of S_k's, $1 \leq k \leq n$, which are positive. Then

$$\lim_{n\to\infty} \operatorname{Prob}\left\{\frac{N_n}{n} < \alpha\right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}}, \quad 0 \leqq \alpha \leqq 1$$

In the proof of P.Erdos and M.Kac, the existence of the mean $E[X_n]$ and variance $E[X_n^2]$, $n = 1, 2, \ldots$, are presumed. We shall extend this result when the mean and variance does not necessary exist. The result is following.

Theorem 1. Let $X_1, X_2, \ldots$ be independent identically distributed random variables and suppose that there exists positive sequence $\{A_n\}$, increasing to infinity such that the distribution of S_n/A_n tends to normal distribution $\Phi(x)$. Then

$$\lim_{n\to\infty} \operatorname{Prob}\left\{\frac{N_n}{n} < \alpha\right\} = \frac{2}{\pi} \arcsin \alpha^{\frac{1}{2}}$$

$$0 \leqq \alpha \leqq 1$$

The key points of the proof are as the same as in the P.Erdos and M.Kac. That is, we shall prove that we can take a particular sequence of independent random variables $G_1, G_2, \ldots$, each having normal distribution $\Phi(x)$, instead of $X_1, X_2, \ldots$. To prove this, we use some known theorems.

Let $-X'$ be independent random variable and has same distribution function $F(x)$ with X, then the distribution function $\widetilde{F}(x)$ of $\widetilde{X} = X + X'$ is $(1 - F(-x)) * F(x)$, $\widetilde{F}(x)$ and $\widetilde{X}$ are said, respectively, the symmetrized distribution of $F(x)$ and the symmetrized random variable X.

Let
$$\Phi_F(h) = \int_{-\infty}^{\infty} \frac{h^2}{x^2 + h^2} \, d F(x)$$
$$\Psi_F(h) = \int_{-\infty}^{\infty} \frac{h^2}{x^2 + h^2} \, d \widetilde{F}(x)$$

which are introduced by K.Kunisawa, and he called typical function and mean concentration function of $F(x)$. In our proof of the theorem, the following Kunisawa's fundamental unequality[2] are used.

Lemma 1. For any $h > 0$

$$(1) \quad 1 - \Psi_{F_1 * \cdots * F_n}(h) \leqq \sum_{k=1}^{n} (1 - F_k(h)),$$

where $F_1, F_2, \ldots, F_n$ are any distribution functions.

Lemma 2. Let α be a positive number ($0 < \alpha < \frac{1}{2}$, $0 < \lambda < 1$) and let $F(x)$ be a distribution function satisfying

$$(2) \quad F(+0) \geqq \lambda > 0, \quad F(-0) \leqq 1 - \lambda, \quad (0 < \lambda < 1)$$

and

$$(3) \quad 1 - F(h) + F(-h) < \alpha,$$

then we have

$$(4) \quad 1 - \Psi_F(h) \geqq K(\alpha, \lambda) \left(1 - \Phi_F(h)\right),$$

where $K(\alpha, \lambda)$ is a positive constant depending on α and λ.

Lemma 3. Under the same assumptions of Theorem 1,

$$\lim_{n\to\infty} n \int_{|x| > \varepsilon A_n} d\widetilde{F}(x) = 0$$

$$\lim_{n\to\infty} \frac{n}{2A_n^2} \int_{|x| < \varepsilon A_n} x^2 \, d\widetilde{F}(x) = 1$$

for every $\varepsilon > 0$, where $\widetilde{F}(x)$ is the symmetrized distribution of $F(x)$ which is distribution of X_k's.