DIFFERENTIAL EQUATIONS WITH INVARIANT PFAFFIAN FORMS

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1. Let Ω be a compact n-dimensional analytic manifold without torsion. We consider a following system of differential equations,

$$(1)\quad \begin{cases} \dfrac{dx_1}{dt} = X_1(x_1, \cdots, x_n), \\ \cdots\cdots \\ \dfrac{dx_n}{dt} = X_n(x_1, \cdots, x_n), \end{cases}$$

where $x_1, \ldots, x_n$ are analytic local coordinates and $X_1, \ldots, X_n$ are one-valued real holomorphic functions in Ω. General solutions of this system can be written down in the following form,

$$x_i = f_i(x_{10}, \cdots, x_{n0}, t), \quad i = 1, \cdots, n,$$

where

$$x_{i0} = f_i(x_{10}, \cdots, x_{n0}, 0), \quad i = 1, \cdots, n,$$

and f_i 's are analytic functions with respect to their arguments.

If we define a transformation S_t by

$$P = S_t P_0,$$
$$P = (x_1, \cdots, x_n), \quad P_0 = (x_{10}, \cdots, x_{n0}),$$
$$x_i = f_i(x_{10}, \cdots, x_{n0}, t), \quad i = 1, \cdots, n,$$

the totality of such transformations forms a one-parameter group. Hence differential equations (1) can be regarded as defining a one-parameter stationary flow S_t in Ω.

We suppose that (1) admits $n-1$ linearly independent (with respect to numerical coefficients) invariant Pfaffian forms (in the sense of E.Cartan)[1],

$$(2)\quad \varpi_i = \sum_{k=1}^{n} A_{ik}(x_1, \cdots, x_n)\, dx_k, \quad i = 1, \cdots, n-1,$$

where A_{ik} 's are one-valued real holomorphic functions in Ω. Then we have

$$(3)\quad \sum_{k=1}^{n} A_{ik} X_k \equiv 0, \quad i = 1, \cdots, n-1 \quad [2]$$

Moreover we assume that ϖ_i 's are exact, i.e.

$$d\varpi_i = 0, \quad i = 1, \cdots, n-1,$$

or, in other words,

$$(4)\quad \frac{\partial A_{ik}}{\partial x_j} = \frac{\partial A_{ij}}{\partial x_k},$$
$$i = 1, \cdots, n-1, \quad j, k = 1, \cdots, n.$$

Under these assumptions, we want to study the behavior of the trajectories of (1). Our main result is the Theorem 3 of § 5 which states the necessary and sufficient condition for every trajectory of (1) to be everywhere dense in Ω. Then we apply this result to the flow in n-dimensional toroid and establish a sufficient condition for the ergodicity of S_t.

2. Let p be a one-dimensional Betti number of Ω, and $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be its independent cycles. We put

$$\int_{\Gamma_k} \varpi_i = \omega_{ik},$$
$$i = 1, \cdots, n-1, \quad k = 1, \cdots, p.$$

Since ϖ_i 's are exact, we can find $n-1$ holomorphic functions $u_1, \ldots, u_{n-1}$ such that

$$du_i = \varpi_i, \quad i = 1, \cdots, n-1.$$

According to the relation (3),

$$\frac{du_i}{dt} = 0, \quad i = 1, \cdots, n-1$$

Hence u_i 's are integrals of (1) and the trajectory of (1) is generally given as an intersection of $n-1$ hypersurfaces

$$du_1 = 0, \quad \cdots, \quad du_{n-1} = 0.$$

u_i 's are, in general, not one-valued since they are additive functions with $\omega_{i1}, \ldots, \omega_{ip}$ as fundamental periods.

We first prove the following

THEOREM 1. If there exist $n-1$ real numbers $\lambda_1, \ldots, \lambda_{n-1}$, not simultaneously zero, such