NOTE ON (LM)-GROUPS OF FINITE ORDERS[*]

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In the present note, we study some properties of a finite group whose lattice of subgroups is lower semi-modular. We, however, use no result of the general theory of lattices.

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NOTATIONS: $S_i(X)=S_{p_i}(X), H_i(X)=H_{p_i}(X)$, $C(X)$, $C_\infty(X)$, $\theta(X)$ and $\Phi(X)$ denote a p_i-Sylow subgroup, a p_i-Sylow complement, the centre, the hypercentre, the commutator subgroup and Φ-subgroup of a group X respectively; (X) may be often omitted. $\mathfrak{N}_Y(X)$ denotes the normalizer of a subgroup X in a group Y.

1. On the p-nilpotency.

DEFINITION 1. A finite group is called p-nilpotent when it has a normal p-Sylow complement.

PROPOSITION 1. Let G be a group whose order has at least three distinct prime factors and let p be one of them. Then G is p-nilpotent if every proper subgroup of G is so.

PROOF. Let G be a group which satisfies our condition. If G is not p-normal in GRUN's sense[(1)], there exist a p-subgroup P and a p-regular element A in G such that A induces a non-identical automorphism into P, by virtue of a theorem of W.BURNSIDE[(2)]. Since $P\cdot\{A\}$ is non-p-nilpotent, we have $G=P\cdot\{A\}$. Let $A=A_1A_2\cdots A_r$ be the Sylow decomposition of A. Then $r\geqq 2$ by our condition. Clearly $G\neq P\cdot\{A_i\}$, whence $P\cdot\{A_i\}=P\times\{A_i\}$. Therefore $G=P\times\{A\}$ which is a contradiction. Hence G is p-normal. Now by a theorem of O.GRÜN[(3)],

$$S_p(G/\theta(G))\cong S_p(\mathfrak{N}(C(S_p))/\theta(\mathfrak{N}(C(S_p))))$$

If $G\neq\mathfrak{N}(C(S_p))$, since the latter is p-nilpotent by our condition, $S_p(\mathfrak{N}(C(S_p))/\theta(\mathfrak{N}(C(S_p))))\neq e$ whence $S_p(G/\theta(G))\neq e$. Therefore, $G\neq\theta(G)$ whence it is easily verified that G is p-nilpotent. If $G=\mathfrak{N}(C(S_p))$ and $S_p\neq C(S_p)$, then induction argument can be applied to $G/C(S_p)$ and we can see that $G/C(S_p)$ is p-nilpotent whence it is easily verified that G is p-nilpotent. Finally if $G=\mathfrak{N}(C(S_p))$ and $S_p=C(S_p)$, then there exists, by a theorem of I.SCHUR[(4)], one H_p in G. Since $G\neq S_p S_q(H_p)$, by our condition, $S_p\cdot S_q(H_p)=S_p\times S_q(H_p)$ whence $G=S_p\times H_p$. Therefore, of course, G is p-nilpotent.

PROPOSITION 2. Let G be a non-p-nilpotent group whose every proper subgroup is p-nilpotent. Then $G=S_p\cdot S_q$ where S_p is normal, $S_q=\{Q\}$ is cyclic, non-normal. And every proper subgroup of G is nilpotent. In particular it is soluble. The converse is also valid.

PROOF. Let G be a group which satisfies our condition. Follow the proof of PROPOSITION 1. First it is evident that the order of is $p^m q^n$ by PROPOSITION 1. Therefore if G is not p-normal, then $A=A_1$, using the same notations as in the proof of PROPOSITION 1, and this proves PROPOSITION 2. Now assume that G is p-normal. Then $\mathfrak{N}(C(S_p))=G$, since if $\mathfrak{N}(C(S_p))\neq G$, G is p-nilpotent, as is easily seen by virtue of the proof of PROPOSITION 1. If $C(S_p)\neq S_p$, induction can be applied to $G/C(S_p)$ and we can easily prove PROPOSITION 2. Finally if $C(S_p)=S_p$, then $G=S_p\cdot S_q$. And if $S_q=T\cup U$ where T and $U\subsetneqq S_q$, since $G\neq S_pT$ and S_pU, $S_p\cdot T=S_p\times T$ and $S_p\cdot U=S_p\times U$ whence $G=S_p\times S_q$ which is a contradiction. Therefore S_q is cyclic and this proves PROPOSITION 2. The converse is obvious.

REMARK 1. Similar results as PROPOSITION 1 and 2 have been obtained by many authors, for instance, O.SCHMIDT[(5)], D.KOLIANKOWSKY[(6)], S.TCHOUNIKHIN[(7)] and K.IWASAWA[(8)]. And our result is a slight modifi-