ON THE RELATIVE STABILITY OF SUMS OF POSITIVE RANDOM VARIABLES

By Tatsuo KAWATA

1. Let $X_1, X_2, \ldots,$ be a sequence of random variables mutually independent. If for a suitable number sequence $\{A_n\}$,

$$(1.1)\qquad \frac{1}{A_n}\sum_{k=1}^{n} X_k$$

tends in probability to 1, we say that the sequence

$$(1.2)\qquad X_1, X_2, \ldots$$

is relatively stable with respect to $\{A_n\}$ and if as $n\to\infty$, X_k/A_n tends in probability to zero uniformly $1\le k\le n$ $\{X_k\}$ is called relatively small. Mr. Bobroff has proved the following theorem.(1)

Theorem 1. *Let $\{X_k\}$ be a sequence of non-negative, mutually independent random variables. If with respect to a number sequence $\{A_n\}$, $A_n>0$, $\{X_k\}$, is relatively stable, then it is relatively small for $\{A_n\}$ and there exists a sequence of positive numbers $\{c_n\}$ such that*

$$(1.3)\qquad \sum_{k=1}^{n}\int_{c_n}^{\infty} dF_k(x)\to 0,$$

$$(1.4)\qquad \frac{1}{c_n}\sum_{k=1}^{n}\int_{0}^{c_n} x\,dF_k(x)\to\infty,$$

where $F_k(x)$ denotes the distribution function of X_k. Conversely if there exists a sequence $\{c_n\}$ satisfying (1.3) and (1.4), then $\{X_k\}$ is relatively stable and relatively small.

Recently K. Kunisawa has given an another simple proof of Theorem 1, with conditions

$$(1.5)\qquad \sum_{k=1}^{\infty}\int_{c_n}^{\infty} dF_k(x)\to 0,$$

$$(1.6)\qquad \sum_{k=1}^{n}\int_{0}^{\infty}\frac{x\,c_n}{x^2+c_n^2}\,dF_k(x)\to\infty,$$

instead of (1.3) and (1.4).

The object of the present paper is to give the conditions for relative stability of $\{X_k\}$ different from the above and to deduce Bobroff's theorem from it. The method is also different from Bobroff's or Kunisawa's and seems to be useful for positive random variables.

2. Lemma 1. *Let $F(x)$ be the distribution function of a random variable X which is non-negative. Then*

$$(2.1)\qquad f(z)=\int_0^{\infty} e^{izx}\,dF(x),\quad z=t+i\tau$$

is analytic in $\tau>0$. $f(t)$ is the characteristic function of X.

This is evident. We say $f(z)$ the analytic characteristic function of X.

Lemma 2. *In order that the non-negative random variable X_k converges in distribution to a variable X, it is necessary and sufficient that the analytic characteristic function $f_k(z)$ of X_k converges to that of X uniformly in every finite closed rectangular domain interior to upper half-plane $\tau=\mathcal{R}z>0$.*

The proof of necessity is quite similar as the ordinary Lévy continuity theorem. We thus prove the sufficiency. Let $f(z)$ be the analytic characteristic function of $X\ (\ge 0)$ and

$$(2.2)\qquad \lim_{n\to\infty}\int_0^{\infty} e^{itx-\tau x}\,dF_n(x)=\int_0^{\infty} e^{itx-\tau x}\,dF(x)$$

uniformly in $-T\le t\le T$, $\tau\ge\tau_0>0$. By the compactness of $\{F_n(x)\}$, there exists a sequence $\{n_i\}$ such that $F_{n_i}(x)\to\varphi(x)$ at continuity points, where $\varphi(x)$ is a non-decreasing function. Then

$$\int_0^{\infty} e^{itx-\tau x}\,dF_n(x)\to\int_0^{\infty} e^{itx-\tau x}\,d\varphi(x).$$

For, taking A so large that $e^{-\tau_0 A}\le\frac{\varepsilon}{2}$, we have

$$\left|\int_A^{\infty} e^{itx-\tau x}\,dF_n(x)\right|\le\int_A^{\infty} e^{-\tau x}\,dF_n(x)\le e^{-\tau_0 A}\le\varepsilon/2,$$

$$\left|\int_A^{\infty} e^{itx-\tau x}\,d\varphi(x)\right|\le\int_A^{\infty} e^{-\tau x}\,d\varphi(x)\le e^{-\tau_0 A}\le\varepsilon/2,$$

and

$$\lim_{n\to\infty}\int_0^{A} e^{itx-\tau x}\,dF_n(x)=\int_0^{A} e^{itx-\tau x}\,d\varphi(x).$$

By (2.2) we get

$$(2.3)\qquad \int_0^{\infty} e^{itx-\tau x}\,d\varphi(x)=\int_0^{\infty} e^{itx-\tau x}\,dF(x).$$