SOME METRICAL THEOREMS ON FUCHSIAN GROUPS

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1. Let G be a Fuchsian group of linear transformations $S_n(z)$ $(n=0,1,2,\ldots)$, which make $|z|<1$ invariant and D_0 be its fundamental domain, which contains $z=0$ and is bounded by orthogonal circles to $|z|=1$ and a closed set e_0 on $|z|=1$. We remark that D_0 can be so constructed that the equivalent points on the boundary of D_0 are equidistant from $z=0$ [1]. Let z_n, D_n, e_n be equivalents of $z_0=0$, D_0, e_0 respectively.

Theorem 1. If $me_0>0$, then $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$. The converse is not true in general.

Proof. Since $me_0>0$, we have $me_n>0$ $(n=0,1,2,\ldots)$. Let

$$u_n(z)=\int_{e_n}\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta\ ,$$

then $u_n(z_n)=u(0)=me_0$, so that

$$me_0=u_n(z_n)\leqq\frac{2\,me_n}{1-|z_n|}\ ,$$

hence

$$\sum_{n=0}^{\infty}(1-|z_n|)\leqq\frac{2\sum_{n=0}^{\infty}me_n}{me_0}\leqq\frac{4\pi}{me_0}<\infty\ .$$

Let $K_1,\ldots,K_n$ $(n\geqq 3)$ be n circles on the w-plane, which lie outside each other. We invert them on any one of them indefinitely, then we obtain infinitely many circles clustering to a non-dense perfect set E. As Myrberg [2] proved, E is of positive logarithmic capacity, so that if we map the outside of E on $|z|<1$ by $w=f(z)$, then $f(z)$ is automorphic to a certain Fuchsian group G, such that $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$. On the other hand, as I have proved in a former paper,[3] $me_0=0$. Hence the converse is not true in general.

2. Let z be any point in $|z|<1$ and (z) be its equivalent in D_0. Let $z=re^{i\theta}$ $(0\leqq r<1)$ be a radius through $e^{i\theta}$. We denote the set $(re^{i\theta})(0\leqq r<1)$ by $E(\theta)$. Then

Theorem 2. (i) If $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, then $\lim_{r\to 1}|(re^{i\theta})|=1$ for almost all $e^{i\theta}$. (ii) If $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, then $E(\theta)$ is everywhere dense in D_0 for almost all $e^{i\theta}$.

I have proved this theorem in a former paper,[4] but my proof depends on a theorem, which is false. A proof is given by Yûjôbô.[5] I will give the following proof, which is somewhat simpler than his. In the proof, we use the following lemma.[6] Let E_0 be a closed set in D_0, which is of positive logarithmic capacity and E_n be its equivalents. We take off $\sum_{n=0}^{\infty}E_n$ from $|z|<1$ and let Δ be the remaining domain. We map Δ on $|\zeta|<1$ and let $\sum_{n=0}^{\infty}E_n$ be mapped on a set e on $|\zeta|=1$. Then

Lemma. (i) If $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, then $0<me<2\pi$. (ii) If $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, then $me=2\pi$.

Proof of Theorem 2.

(i) Suppose that $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, then the Green's function

$$G(z)=\sum_{n=0}^{\infty}\log\left|\frac{1-\bar{z}_n z}{z-z_n}\right| \qquad (1)$$

exists and $\lim_{z\to e^{i\theta}}G(z)=0$ almost everywhere on $|z|=1$, when $z\to e^{i\theta}$ non-tangentially to $|z|=1$. From this, we see that $\lim_{r\to 1}|(re^{i\theta})|=1$ for almost all $e^{i\theta}$.

(ii) Next suppose that $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$. Let K_0 be a disc contained in D_0 and K_n be its equivalents and C_n be its boundary. We take off $\sum_{\nu=0}^{\infty}K_\nu$ from $|z|<1$ and Δ be the remaining domain and we take off $\sum_{\nu=0}^{\infty}K_\nu$ from $|z|<1$ and Δ_n be the remaining domain.

Let $u_n(z)$ be a bounded harmonic function in Δ_n, such that

$$\left.\begin{aligned} u_n(z)&=0 \quad \text{on} \quad \sum_{\nu=0}^{n}C_\nu\ ,\\ u_n(z)&=1 \quad \text{on} \quad |z|=1\ .\end{aligned}\right\} \qquad (2)$$

First we will prove that

$$\lim_{n\to\infty}u_n(z)\equiv 0 \quad \text{in}\ \Delta \qquad (3)$$

We map Δ on $|\zeta|<1$, then by the lemma, $|z|=1$ is mapped on a null set on $|\zeta|=1$. Let $\sum_{\nu=n+1}^{\infty}C_\nu$ be mapped on a set e_n on $|\zeta|=1$, then $\lim_{n\to\infty}me_n=0$. Let $u_n(z)$ become a harmonic function $v_n(\zeta)$ in $|\zeta|<1$, then, since $|z|=1$ is mapped on a null set, we have