ON UNIFORMIZING FUNCTIONS

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§ 1. Let S be a closed Riemann surface with genus $g \geqq 2$, whose equation is given by $S(x, y) = 0$, $S(x, y)$ being an irreducible polynomial of degree n in x and m in y. Let $f(t)$ and $g(t)$ be meromorphic functions in the circle $|t| < R$. If $S(f(t), g(t)) \equiv 0$, we say that $f(t)$ and $g(t)$ are uniformizing functions. In this note we will prove the following theorem:

$$\varlimsup_{r \to R} \frac{T(r, f)}{\log \frac{1}{R-r}} \leqq \frac{m}{2g-2},$$

$$\varlimsup_{r \to R} \frac{T(r, g)}{\log \frac{1}{R-r}} \leqq \frac{n}{2g-2},$$

where $T(r, f)$ and $T(r, g)$ are Nevanlinna's characteristic functions.

§ 2. The algebraic function can be uniformized by means of Fuchsian functions $x = x(z)$, $y = y(z)$, in such a manner that in a sufficiently small neighbourhood of a point z in the principal circle of the group the correspondence between the points of the plane and the points of S is one to one.

Putting $z = z(x, y)$ and $u = \log \frac{\left|\frac{dz}{dx}\right|}{1-|z|^2}$, then $\Delta_x u = 4e^{2u}$, where $\Delta_x u = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}$ and $x = \xi + i\eta$. Putting $z(t) = z(f(t), g(t))$ and $u(t) = \log \frac{\left|\frac{dz}{dx}\right|\left|\frac{dx}{dt}\right|}{1-|z|^2} = \log W$, then $u(t) = u + \log\left|\frac{dx}{dt}\right|$ and $\Delta u(t) = 4e^{2u(t)}$.

In this note, by infinity points on S we mean points where $x = \infty$. We suppose that infinity points on S are not branch points.

(I). At a branch point $(x, y) = (a, b)$ of order $m-1$:

$$y - b = a_p (x-a)^{p/m} + a_{p+1}(x-a)^{(p+1)/m} + \cdots,$$

where $a_p \neq 0$;

$$(1) \qquad u = (1-m)/m \log|x-a| + v(x),$$

where $v(x)$ is bounded function in a neighborhood of the point (a, b). Since $f(t)$ and $g(t)$ are single-valued functions,

$$(2) \qquad f(t) - a = a_{km}(t-t_0)^{km} + a_{km+1}(t-t_0)^{km+1} \cdots,$$

$$g(t) - b = b_{kp}(t-t_0)^{kp} + b_{kp+1}(t-t_0)^{kp+1} \cdots,$$

where $f(t_0) = a$, $g(t_0) = b$, $a_{km} \neq 0$, $b_{kp} \neq 0$. From (1) and (2) we get $W = |t-t_0|^{k-1} S(t)$, where $S(t)$ is a bounded function in a neighbourhood of $t = t_0$.

(II). At an infinity point on S:

$$(3) \qquad u = -2\log|x| + v(x),$$

$$(4) \qquad f(t) = \frac{c_{-k}}{(t-t_0)^k} + \frac{c_{-k+1}}{(t-t_0)^{k-1}} + \cdots,$$

where $c_k \neq 0$. From (3) and (4) we get

$$W = |t-t_0|^{k-1} S(t).$$

By Ahlfors' theorem (An extension of Schwarz's lemma, Trans. of Amer. Math. Soc. 43, 1938) we get the following theorem:

Theorem 1. We suppose that infinity points on S are not branch points. If we put $z(t) = z(f(t), g(t))$, then

$$\frac{\left|\frac{dz}{dx}\right|\left|\frac{dx}{dt}\right|}{1-|z|^2} \leqq \frac{R^2}{R^2-|t|^2}.$$

§ 3. Let $G(x, y, \Gamma_1, \Gamma_2)$ be harmonic on S, except two logarithmic singular points at Γ_1 and Γ_2 and let $\Gamma_1(c_1, d_1)$ and $\Gamma_2(c_2, d_2)$ be branch points of order $k_1 - 1$ and $k_2 - 1$, respectively;

$$G(x, y; \Gamma_1, \Gamma_2) = \begin{cases} \frac{1}{k_1}\log\frac{1}{|x-c_1|} + \text{bounded harmonic function at } \Gamma_1, \\ \frac{1}{k_2}\log|x-c_2| + \text{bounded harmonic function at } \Gamma_2. \end{cases}$$

We put $G(t, \alpha, \Gamma) = G(f(t), g(t); \alpha, \Gamma)$ and $\overline{m}(r, \alpha, \Gamma) = \frac{1}{2\pi}\int_0^{2\pi} G^+(t, \alpha, \Gamma)\, d\theta$, where $t = re^{i\theta}$. Putting

$$G(t; \alpha_1, \alpha_2) = G^+(t; \alpha_1, \Gamma) - G^+(t; \alpha_2, \Gamma) + U(t, \alpha_1, \alpha_2, \Gamma),$$

$$G^+(t; \alpha, \Gamma_1) - G^+(t; \alpha, \Gamma_2) = U(t, \alpha, \Gamma_1, \Gamma_2),$$

then $U(t, \alpha_1, \alpha_2, \Gamma)$ and $U(t; \alpha, \Gamma_1, \Gamma_2)$ are bounded functions in the circle $|t| < R$. Hence we get