REPRESENTATION OF FUNCTIONS ANALYTIC IN A MULTIPLY-CONNECTED DOMAIN

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1. We may and do use, as a canonical domain of multiplicity $n\ (>2)$, a concentric annular ring slit along concentric circular arcs. Let the boundary components of such a domain D, laid on z-plane, be

$$C_1:\ |z|=1; \qquad C_2:\ |z|=Q\ (<1);$$

$$C_j:\ |z|=m_j, \quad \theta_j \leqq \arg z \leqq \theta_j+\tau_j \quad (3 \leqq j \leqq n),$$

and the interior and the exterior sides of the slits $C_j\ (3 \leqq j \leqq n)$ be

$$C_j^{(i)}:\ |z|=m_j-0, \quad \theta_j \leqq \arg z \leqq \theta_j+\tau_j,$$

$$C_j^{(e)}:\ |z|=m_j+0, \quad \theta_j+\tau_j \geqq \arg z \geqq \theta_j,$$

respectively. The total boundary of D be denoted by

$$C=\sum_{j=1}^{n} C_j .$$

Any function $U(z)$ regular harmonic in the domain D and continuous on the closed domain $D+C$ is represented by Green's formula in the form

$$U(z)=\frac{1}{2\pi}\int_C U(\zeta)\frac{\partial g(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta,$$

$g(\zeta; z)$ being, as usual, Green function (with variable ζ) of D with singularity at z, ν_ζ and s_ζ denoting inward normal and arc-length parameter at a boundary point ζ.

If we denote the equation of the boundary C by $z=\zeta(s)$ and the harmonic measure of a part of C from a fixed point to the point $\zeta(s)$ by $\omega(z, \zeta(s))$, then we have

$$\frac{1}{2\pi}\frac{\partial g(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta = d\omega(z,\zeta(s)) \equiv \omega(z, d\zeta(s)).$$

But, we use here an another aggregation, namely the one corresponding to Herglotz type. Let $\Phi(z)$ be an analytic function one-valued and regular in D and continuous on $D+C$. We denote by $G(\zeta, z)$ an analytic function of z whose real part coincides with $g(\zeta, z)$; $G(\zeta, z)$ being uniquely determined except an additive purely imaginary quantity depending possibly on ζ and possessing multi-valuedness due to periodicity moduli with respect to the boundary components. We have then, by the formula mentioned above,

$$\Phi(z)=\frac{1}{2\pi}\int_C \mathcal{R}\Phi(\zeta)\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta + ic,$$

c being a real constant.

We now assume that $\mathcal{R}\Phi(z)$ is of bounded variation along C. Then, so is also the function $(\zeta \in C_j)$

$$\rho_j(\varphi)=\int^{\varphi} \mathcal{R}\Phi(\zeta)\,ds_\zeta \qquad (\varphi=\arg\zeta);$$

in fact,

$$\int_{C_j} |d\rho_j(\varphi)| = \int_{C_j} |\mathcal{R}\Phi(\zeta)|\,ds_\zeta .$$

In this case, we may write the expression as in the Herglotz type which states

$$\Phi(z)=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{C_j}\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,d\rho_j(\varphi)+ic.$$

Now, considering residue at point z, we have particularly

$$\frac{1}{2\pi}\int_C \frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta = 1,$$

and hence

$$1=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{C_j}\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,d\sigma_j(\varphi),$$

where $\sigma_j(\varphi)$ is defined by

$$\sigma_j(\varphi)=\begin{cases} \varphi & \text{on } C_1, \\ -Q\varphi & \text{on } C_2; \\ m_j(\varphi-\theta_j) & \text{on } C_j^{(i)}, \\ -m_j(\varphi-\theta_j-\tau_j) & \text{on } C_j^{(e)} \end{cases} \quad (3 \leqq j \leqq n).$$

The last equation shows that an additive purely imaginary constant ic contained in the general representation vanishes out for the particular function $\Phi(z) \equiv 1$.

2. Consider now an analytic function $f(z)$ one-valued and regular in D and piecewise regular on $D+C$.