ON GAP SERIES

By Tatsuo Kawata

1. Let $\varphi(x)$ be a continuous periodic function with period 2π which satisfies the Lipschitz condition of order α $(0<\alpha<1)$. Suppose through this paper that

$$(1.1) \qquad \int_0^{2\pi} \varphi(x)\,dx = 0 .$$

The object of the present paper is to discuss the various similar properties of the series

$$(1.2) \qquad \sum_{n=1}^{\infty} c_n \varphi(\lambda_n x),$$

as the Fourier series with gaps, for example, the convergence, mean convergence, absolute convergence of (1.2) and distribution properties of partial sums of (1.2).

Among other results, M.Kac has proved that if λ_n are positive integers such that

$$(1.3) \qquad \frac{\lambda_{n+1}}{\lambda_n} \geqq q > 1 ,$$

then the convergence of

$$(1.4) \qquad \sum_{1}^{\infty} c_n^2$$

implies the convergence of (1.2) at almost all x and the mean convergence in every finite interval.[1] Recently M.Udagawa and the author proved that the convergence property of (1.2) under the condition $\sum c_n^2 < \infty$ holds good for non-integer sequence λ_n. Also it was shown by M.Kac that, if $\varphi(x) = e^{ix}$, then the above result also holds even if the integral character is not supposed [2], and in this case the divergence of (1.4) implies the almost everywhere divergence of (1.2). The last fact is due to M. Kac [3] and P.Hartman [4]. For general series (1.2), the more severe condition than (1.3) on gaps is necessary for the validity of the last fact. Recently M.Udagawa and the author proved that the almost everywhere convergence of (1.2) under the condition $\sum c_n^2 < \infty$ follows for non-integer sequence $\{\lambda_n\}$ with (1.3). In § 2, we shall prove the more complete theorem (Theorem 1) as to (1.2) which is well known for Fourier series with gaps (1.3), under the following gap condition,

$$(1.5) \qquad \frac{\lambda_{n+1}}{\lambda_n} \geqq n^c$$

where c is any positive number, and the λ_n is not necessarily an integer.

The Fourier series with gaps (1.3) of a bounded function, converges absolutely. This is well known theorem of S.Sidon, which was generalized to the non-harmonic series, (almost periodic Fourier series) by M.Udagawa and the author. [5] Corresponding theorem for the general series (1.2) will be shown in Theorem 12 .

In the last section we shall consider the behavior of the distribution of partial sums

$$(1.6) \qquad \sum_{k=1}^{n} c_k \varphi(\lambda_k x) = S_n(x).$$

2. Lemma 1. Let $\varphi(x)$ belong to $Lip\,\alpha$ $(0<\alpha\leqq 1)$ satisfying (1.1) and let (1.3) to be hold. We put

$$(2.1) \qquad \sigma(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\sin^2 x/2}{x^2/2}\,dx$$

Then

$$(2.2) \qquad \left|\int_{-\infty}^{\infty} \varphi(\lambda_j x)\varphi(\lambda_k x)\,d\sigma(x)\right| \leqq A q^{-\alpha|j-k|}$$

where A is a constant independent of j and k .

This lemma was proved by M.Kac [6] in the case $\{\lambda_k\}$ are integers and was generalized by M.Udagawa and the author to general case.[7] We shall suppose $\varphi(x)$ to be real in this paper.

Lemma 2. Let a_μ, b_μ $(\mu=1,2,\cdots)$ be the Fourier cosine and sine coefficients of $\varphi(x)$ which satisfies the conditions in Lemma 1 ($a_0=0$ by (1.1)), and denote

$$(2.3) \qquad B = \frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu^2+b_\mu^2)$$

If (1.3) holds and

$$(2.4) \qquad q^{\alpha}-1 > \frac{2A}{B},$$

A being a constant in (2.2), then

$$(2.5) \qquad \int_{-\infty}^{\infty} S_n^2(x)\,dx \geqq \Delta \sum_{k=1}^{n} c_k^2 ,$$

where

$$(2.6) \qquad S_n(x) = \sum_{k=1}^{n} c_k\,\varphi(\lambda_k x)$$

and

$$\Delta = B - 2AC_q, \quad C_q = \frac{1}{q^{\alpha}-1} .$$

Proof. We have

$$\int_{-\infty}^{\infty} S_n^2(x)\,d\sigma(x) = \int_{-\infty}^{\infty}\sum_{k,j=1}^{n} c_k\varphi(\lambda_k x)c_j\varphi(\lambda_j x)\,d\sigma(x)$$

$$= \int_{-\infty}^{\infty}\sum_{k=1}^{n} c_k^2\varphi^2(\lambda_k x)\,d\sigma(x)$$

$$(2.7) \qquad + 2\sum_{k\neq j}\int_{-\infty}^{\infty} c_k c_j\,\varphi(\lambda_k x)\varphi(\lambda_j x)\,d\sigma(x).$$