ON A CHARACTERIZATION OF MULTIPLE NORMAL DISTRIBUTIONS

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Let the n-dimensional multiple distribution defined by n random variables x_1, ..., x_n be the normal distribution with the probability density

$$(1)\quad f(x_1,\ldots,x_n) = (2\pi)^{-\frac{n}{2}}\sigma^{-n} \cdot \exp\left(-\frac{\sigma^2}{2}\sum_{k=1}^{n}(x_k - m_k)^2\right).$$

Then the two random variables y, z defined by the linear forms:

$$(2)\quad y = \sum_{k=1}^{n}\alpha_k x_k,\quad z = \sum_{k=1}^{n}\beta_k x_k$$

are independent if and only if

$$(3)\quad \sum_{k=1}^{n}\alpha_k\beta_k = 0$$

holds. Now we consider the converse of this property.

Theorem. *Let $x_1, \ldots, x_n$ be n random variables. If any two random variables y, z defined as the linear forms of $x_1, \ldots, x_n$ by (2) are independent whenever the relation (3) holds, then the multiple distribution of $x_1, \ldots, x_n$ is the normal distribution with the probability density (1).*

Proof. Let $\varphi_\kappa(s) = E(\exp(is\,x_\kappa))$ ($\kappa = 1, \ldots, n$) be the characteristic function of x_k. Since $y = x_1\cos\theta + x_2\sin\theta$ and $z = -x_1\sin\theta + x_2\cos\theta$ are independent by our hypothesis, we have

$$(4)\quad E(\exp(is\,y + it\,z)) = E(\exp(is\,y))\,E(\exp(it\,z)).$$

Since x_1 and x_2 are independent by our hypothesis, we can represent the both side of (4) by φ_1 and φ_2 :

$$(5)\quad \varphi_1(s\cos\theta - t\sin\theta)\,\varphi_2(s\sin\theta + t\cos\theta) = \varphi_1(s\cos\theta)\,\varphi_1(-t\cdot\sin\theta)\,\varphi_2(s\sin\theta)\,\varphi_2(t\cos\theta).$$

Putting $\theta = \pi/4$ and $\sqrt{2}\,s$, $\sqrt{2}\,t$ instead of s, t in (5) we have

$$(6)\quad \varphi_1(s-t)\,\varphi_2(s+t) = \varphi_1(s)\,\varphi_1(-t)\,\varphi_2(s)\,\varphi_2(t).$$

Taking $s = t$ or $s = -t$ in (6), we have especially the relations:

$$(7)\quad \varphi_2(2s) = |\varphi_1(s)|^2\cdot\varphi_2(s)^2,\qquad \varphi_1(2s) = \varphi_1(s)^2\cdot|\varphi_2(s)|^2.$$

Now follows from (7)

$$(8)\quad |\varphi_1(s)| = |\varphi_2(s)|$$

Hence we have also from (7) $|\varphi_\kappa(2t)| = |\varphi_\kappa(t)|^4$ (k=1,2). Now put $s = rt$ ($r = 1,2,\ldots$) in (6) we have then $|\varphi_\kappa((r-1)t)| \cdot |\varphi_\kappa((r+1)t)| = |\varphi_\kappa(rt)|^2\cdot|\varphi_\kappa(t)|^2$. Thus we can prove by the mathematical induction the relation $|\varphi_\kappa(rt)| = |\varphi_\kappa(t)|^{r^2}$ ($r = 1,2,\ldots$). Take then $t = s/p$, we have

$$(9)\quad |\varphi_\kappa(\lambda s)| = |\varphi_\kappa(s)|^{\lambda^2}$$

for $\lambda = r/p$. This relation holds also for any positive number λ by the continuity of φ_κ .

Let us put s=1 in (9), we have $|\varphi_\kappa(\lambda)| = |\varphi_\kappa(1)|^{\lambda^2} = \exp(\alpha_\kappa\lambda^2)$, $\alpha_\kappa \leqq 0$. By the relation (8) we have $\alpha_1 = \alpha_2 = \alpha$. Hence φ_κ has the functional form:

$$\varphi_\kappa(t) = \exp(\alpha t + 2\pi i\,\theta_\kappa(t)).$$

From $\varphi_\kappa(-t) = \overline{\varphi_\kappa(t)}$ follows $\theta_\kappa(-t) \equiv \theta_\kappa(t)$ (mod.1). Since we have the relation $\varphi_\kappa(2t) = \varphi_\kappa(t)^3\,\varphi_\kappa(-t)$ from (7), (8), $\theta_\kappa(t)$ must satisfy $\theta_\kappa(2t) \equiv 2\,\theta_\kappa(t)$ (mod.1). From this follows also $\theta_\kappa(2^r t) \equiv 2^r\theta_\kappa(t)$ (mod.1). Now choose an irrational number ω and put $\theta_\kappa(\omega)$ $m_\kappa\omega$ (mod.1). Then we have

$$(10)\quad \theta_\kappa(\lambda) \equiv m_\kappa\lambda \quad (\text{mod.}1)$$

for $\lambda = 2^r\omega$. Since $\theta_\kappa(t)$ is continuous and the relation (10) holds for a dense subset $\{2^r\omega\ ;\ r = 1,2,\ldots\}$, (10) holds also for every value λ .