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A COHOMOLOGICAL APPROACH TO THEORY OF GROUPS OF PRIME POWER ORDER

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§0. Introduction

Let G be a finite group and A a G-module. Consider the set of group extensions

$$(\Gamma) \qquad 0 \to A \to \Gamma \to G \to 1$$

in which the G-module of A defined via conjugation of G coincides with the one already given in A. Two extensions (Γ) and (Γ') are said to be equivalent if there exists a homomorphism $f : \Gamma \to \Gamma'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & \Gamma \arrow[r, "j"] \arrow[d, "f"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r, "i'"] & \Gamma' \arrow[r, "j'"] & G \arrow[r] & 1
\end{tikzcd}$$

is commutative.

Let $\mathcal{E}(\mathcal{G}, \mathcal{A})$ be the set of equivalence classes of such extensions. It is well-known that there exists a natural 1-1 correspondence

$$H^2(G, A) \xleftrightarrow{\theta} \mathcal{E}(G, A)$$

with $\theta[\Gamma]$ the factor set of the extension (Γ). Good description of $H^2(G, A)$ is then an effective tool to the study of group extensions of A by G. This material has been used by several authors: Babakhanian [1], Baer [2], Beyl [4], Evens [7], Gruenberg [9], Schreier [20] [21], Stammbach [22] ... to obtain group theoretical results.

In this work, we restrict ourselves to the case where G is a group of prime power order (i.e. a p-group); in such a case, A can be chosen to be central and elementary. Our method is focussed on the Hochschild-Serre spectral sequence of a central extension: by studying the relation between the Hochschild-Serre filtration of $H^2(G, A)$ and the Frattini class of Γ, we obtain cohomological proofs of results concerning the Frattini subgroup of a p-group. Most of these results were already proved by other group theorists (Berger-Kovaćs-Newman [3], Blackburn [5], Kahn [13] [14], Hobby [10], Thompson [23] ...).

This note is organized as follows. In §1, we consider the central extension by an elementary abelian p-group and the term $E_\infty^{i,j}$, $(i+j = 2)$ of the Hochschild-Serre spectral sequence for it. §2 is devoted to the study of the relation between the Hochschild-Serre filtration of $H^2(G, A)$ and the Frattini class of Γ; the main results of this section are Theorems 2.1 and 2.3. They are applied to the study of p-groups with cyclic Frattini

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