M. ITOH
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CONFORMAL GEOMETRY OF RICCI FLAT 4-MANIFOLDS

Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

BY MITSUHIRO ITOH

Introduction

Let M be a compact, connected oriented 4-manifold. Let g be a smooth Riemannian metric on X. Then the Riemannian curvature tensor $R=R_{ijkl}$ is defined naturally, and the Ricci tensor $Ric=R_{ij}$ and the scalar curvature R_g by the trace of the Riemannian curvature tensor and the trace of the Ricci tensor, respectively.

Further we define the Weyl conformal tensor $W=W_{ijkl}$ by a linear combination of $R=R_{ijkl}$, Ric and R_g in such a way that the tensor W is invariant under a conformal change of metrics.

We will investigate in this paper the moduli $\mathcal{E}(M)$ of Ricci flat metrics on certain 4-manifolds M from 4-dimensional conformal geometry. Here we mean by the moduli the space of all Ricci flat metrics of volume one modulo diffeomorphisms of M.

Since a Ricci flat metric is Einstein, the moduli is considered naturally as the moduli of Einstein metrics of $R_g=0$.

We have indeed the following premoduli theorem due to K. Koiso ([16] and [5]).

Given an Einstein metric g. Then there is a finite dimensional real analytic submanifold $\mathcal{Z}$ in a slice $\mathcal{S}$ at g such that (i) $g\in\mathcal{Z}$, (ii) $T_g\mathcal{Z}$ coincides with the space of infinitesimal Einstein deformations and (iii) the intersection $\mathcal{E}(M)\cap\mathcal{S}$, called the premoduli around g, is a real analytic subvariety of $\mathcal{Z}$.

We restrict ourself to Ricci flat 4-manifolds having topological invariant $\chi+(3/2)\tau=0$, more precisely, manifolds whose universal covering is a K3 surface. We can then apply the Torelli type theorem for K3 surfaces together with the notion of anti-self-dual conformal structure and get a complete description of manifold structure of $\mathcal{E}(M)$.

By applying the Chern-Weil theorem for characteristic classes one has the following identity which is valid for an arbitrary Riemannian 4-manifold (M, g)

$$\chi(M)+\frac{3}{2}\tau(M)=\frac{1}{48\pi^2}\int_M\{R_g^2-3|Ric(g)|^2\}dv_g+\frac{1}{4\pi^2}\int_M|W^+(g)|^2dv_g$$

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