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VOLUME-PRESERVING GEODESIC SYMMETRIES ON FOUR-DIMENSIONAL 2-STEIN SPACES

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1. Introduction

Let (M, g) be an n-dimensional Riemannian manifold such that all local geodesic symmetries are volume-preserving (up to sign). Locally symmetric spaces, naturally reductive homogeneous spaces and commutative spaces are examples of such manifolds. (See [14] for more details.) To our knowledge an example which is not locally homogeneous is not known and there is some support for an affirmative answer to the following question:

Are Riemannian manifolds such that all local geodesic symmetries are volume-preserving, locally homogeneous manifolds?

For two- and three-dimensional manifolds this is indeed the case [10], but for $n \geqq 4$, it is still an open problem.

In [12], we started our research about this question for four-dimensional manifolds and proved that four-dimensional *Kähler* manifolds with volume-preserving geodesic symmetries are locally symmetric. (See also [11].) On the other hand, a well-known theorem of G. R. Jensen [8] states that any four-dimensional locally homogeneous Einstein space is locally symmetric. In view of this result, it is worthwhile to consider the open problem for the class of *Einstein* spaces.

In this paper we study this problem for a particular subclass of Einstein spaces, namely the so-called *2-stein spaces* [2]. In this way we answer the problem stated at the end of [12]. More precisely, we prove:

MAIN THEOREM. *Let (M, g) be a connected four-dimensional 2-stein space with volume-preserving local geodesic symmetries. Then (M, g) is locally flat or locally isometric to a rank one symmetric space.*

This theorem is a generalization of the theorem of Lichnerowicz and Walker about four-dimensional harmonic spaces (see for example [1], p. 166).

The paper is organized as follows. In section 2 we first consider general Riemannian manifolds with volume-preserving local geodesic symmetries. Then, in section 3, we write down some useful facts about the special geometry of four-dimensional Einstein spaces. Finally, the proof of the Main Theorem is

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