NOTE ON IRREDUCIBLE DECOMPOSITION OF A POSITIVE LINEAR FUNCTIONAL

By Hisaharu UMEGAKI

In this paper we shall introduce a stationary natural mapping in W^*-algebra generated by a two-sided representation of a D^*-algebra $\mathfrak{A}$ with a motion G (e.g. cf. [8]) — a D^*-algebra $\mathfrak{A}$ is mean by a normed*-algebra with an approximate identity and a motion G is mean by a group of $*$automorphisms on $\mathfrak{A}$ (the motion has been introduced by Segal for C^*-algebra). Next, applying the stationary natural mapping and the decomposition theorem of Segal (cf. Th.4 and its proof of [7]) we shall prove an ergodic decomposition of a G-stationary semi-trace of separable $\mathfrak{A}$ under a restriction which generalizes an irreducible decomposition of finite semi-trace (cf. Th.1 of [9], I), ergodic decomposition of G-stationary trace (cf. Th.6 of [8]) and ergodic decomposition of invariant regular measure on a compact metric space with a group of homeomorphisms (cf. Th. in App. II of [3] and Th.7 of [7]).

1.[0] Let $\mathfrak{A}$ be a D^*-algebra with an approximate identity $\{e_\alpha\}_{\alpha \in D}$ and with a motion $G\,(=\{s\})$ i.e. D is a directed set and $e_\alpha^* = e_\alpha$, $\|e_\alpha\| \leq 1$ for all $\alpha \in D$, $\|e_\alpha x - x\| \to 0$ for all $x \in \mathfrak{A}$, and any $s, t \in G$ are automorphisms on $\mathfrak{A}$ such that $\|x^s\| = \|x\|$, $x^{s*} = x^{*s}$ and $(x^s)^t = x^{st}$ for all $x \in \mathfrak{A}$. Let τ be a G-stationary semi-trace of $\mathfrak{A}$, i.e. τ is a linear functional on the self-adjoint subalgebra generated by $\{xy;\ x, y \in \mathfrak{A}\}$ (i.e. $\mathfrak{A}^2$) such that $\tau(x^*x) \geq 0$, $\tau(yx) = \tau(xy) = \overline{\tau(y^*x^*)}$, $\tau((e_\alpha x)^* e_\alpha x) \xrightarrow[\alpha]{} \tau(x^*x)$, $\tau((xy)^*(xy)) \leq \|x\|^2 \tau(y^*y)$ and $\tau(x^s y^s) = \tau(xy)$ for all $x, y \in \mathfrak{A}$ and $s \in G$.

Putting $\mathfrak{N} = \{x \in \mathfrak{A};\ \tau(x^*x) = 0\}$, $\mathfrak{N}$ is a two-sided ideal in $\mathfrak{A}$. Let $\mathfrak{A}^\theta$ be qoutient algebra of $\mathfrak{A}\,(= \mathfrak{A}/\mathfrak{N})$ and for any $x \in \mathfrak{A}$ let x^θ be the class containing x. Letting $(x^\theta, y^\theta) = \tau(y^*x)$ for all $x, y \in \mathfrak{A}$, $\mathfrak{A}^\theta$ is an incomplete Hilbert space. Let $\mathfrak{H}$ be competion of $\mathfrak{A}^\theta$. Putting $x^a y^\theta = (xy)^\theta$, $x^b y^\theta = (yx)^\theta$ and $j y^\theta = y^{*\theta}$ for all $x, y \in \mathfrak{A}$, $\{x^a, x^b, j, \mathfrak{H}\}$ defines a two-sided representation of $\mathfrak{A}$. Moreover putting $U_s y^\theta = (y^s)^\theta$ for all $s \in G$ and $y \in \mathfrak{A}$, $\{U_s, \mathfrak{H}\}$ is a dual unitary representation of G. For, $(U_s y^\theta, x^\theta) = (y^{s\theta}, x^\theta) = \tau(x^* y^s) = \tau(x^{s^{-1}*} y) = (y^\theta, U_{s^{-1}} x^\theta)$ and $U_{st} y^\theta = (y^{st})^\theta = U_t y^{s\theta} = U_t U_s y^\theta$. Then we have:

(1) $(x^s)^a = U_s x^a U_{s^{-1}}$ and $(x^s)^b = U_s x^b U_{s^{-1}}$ for all $x \in \mathfrak{A}$ and $s \in G$.

For, $U_s x^a U_{s^{-1}} y^\theta = U_s x^a (y^{s^{-1}})^\theta = U_s (x y^{s^{-1}})^\theta = (x^s y)^\theta = x^{sa} y^\theta$ and similarly for the latter. Putting W^a, W^b and W_G W^*-algebras generated by $\{x^a,\ x \in \mathfrak{A}\}$, $\{x^b;\ x \in \mathfrak{A}\}$ and $\{U_s,\ s \in G\}$ respectively, $W^a = W^{b\prime}$, $W^{a\prime} = W^b$, $jAj = A^*$ for all $A \in W^a \cap W^b$ and the τ is G-ergodic[1] if and only if $W^a \cap W^b \cap W_G' = \{\lambda I\}$ (cf. Th.2 and Th.5 of [8]) where for any set F of bounded operators on $\mathfrak{H}$ F' is the commutor of F.

Let $\mathfrak{L}$ be the family of all bounded elements v in $\mathfrak{H}$ (i.e. v belongs to $\mathfrak{L}$ if and only if $\|x^b v\| \leq M\|x^\theta\|$ for all cf. [8] and [9]) whose corresponding bounded operators on $\mathfrak{H}$ be v^a and v^b such that $v^a x^\theta = x^b v^\theta$, $v^b x^\theta = x^a v$. Then $\{x^\theta;\ x \in \mathfrak{A}\} \subset \mathfrak{L}$ and $x^{\theta a} = x^a$ for all $x \in \mathfrak{A}$, and the following relations are equivalent each other: for any v_1 and v_2 in $\mathfrak{L}$ $v_1^a = v_2^a$, $v_1^b = v_2^b$ (both as operator) and $v_1 = v_2$ (as point in $\mathfrak{H}$). Now we can define in $\mathfrak{L}$ a $*$-involution and a ring product: v^* and $v_1 v_2\,(= v_1^a v_2 = v_2^b v_1)$ for all $v, v_1, v_2 \in \mathfrak{L}$ satisfying that $v^* = jv$, $v^{*a} = v^{a*}$, $v^{*b} = v^{b*}$ (v^{a*}, v^{b*} are adjoint operators of v^a and v^b), $j v^a j = v^{b*}$, $(v_1 v_2)^a = v_1^a v_2^a$, $(v_1 v_2)^b = v_2^b v_1^b$ and $(\lambda_1 v_1 + \lambda_2 v_2)^d = \lambda_1 v_1^d + \lambda_2 v_2^d$ (for $d = a$ or b) (cf. p.35 of [8], p.61 of [9], II).

(2) $U_s v \in \mathfrak{L}$ and $(U_s v)^a = U_s v^a U_{s^{-1}}$,