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A closed form for unitons

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1. Introduction

Harmonic maps between Riemannian manifolds M and N are critical values of an energy functional

$$\text{energy}(S : M \to N) = \frac{1}{2}\int_M |dS|^2.$$

In the case of surfaces in $\mathrm{U}(N)$, with the standard (bi-invariant) metric, the energy takes the form

$$\text{energy}(S) = \frac{1}{16\pi}\int_{\boldsymbol{R}^2}\left(|S^{-1}\frac{\partial}{\partial x}S|^2 + |S^{-1}\frac{\partial}{\partial y}S|^2\right)dx \wedge dy. \tag{1.1}$$

Unitons are harmonic maps $S : \boldsymbol{S}^2 \to \mathrm{U}(N)$. We write $\mathrm{Harm}(\boldsymbol{S}^2, \mathrm{U}(N))$ for the space of unitons. Some authors call them multi-unitons.

We are concerned with the based maps

$$\mathrm{Harm}_k^*(\boldsymbol{S}^2, \mathrm{U}(N)) \stackrel{\text{def}}{=} \{S \in \mathrm{Harm}(\boldsymbol{S}^2, \mathrm{U}(N)) : S(\infty) = \boldsymbol{I}, \text{energy}(S) = k\}.$$

In [**Uhl**], Uhlenbeck showed that all unitons could be constructed from simpler unitons by 'adding a uniton'. This construction was investigated from different perspectives by Wood, Valli, Guest, Ohnita and Segal. We approach the question of constructing unitons via algebraic integration, using a twistor construction of Hitchin and Ward ([**Hi**], [**Wa**]).

We proved in [**An1**] that the based unitons, $\mathrm{Harm}^*(\boldsymbol{S}^2, \mathrm{U}(N))$, are isomorphic to uniton bundles, with energy corresponding to the bundles' second Chern class. In this paper we apply Horrocks' monad construction to the uniton bundles.

THEOREM A. *Based, rank-N unitons of energy $k/2$ are all of the form*

$$S = \boldsymbol{I} + a\alpha_2^{-1}(\alpha_1 + x\alpha_2 + iy\boldsymbol{I})^{-1}b$$

for some choice of $N \times k$, $k \times N$, $k \times k$ and $k \times k$ matrices, a, b, α_1, α_2. (Multiplication is matrix multiplication.)

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