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Complex bordism of the dihedral group

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Introduction.

Let G be a finite group. By a G-U-manifold we mean a weakly complex manifold with a free G-action preserving its weakly complex structure. The group of bordism classes of closed G-U-manifolds is isomorphic to the complex bordism group $MU_*(BG)$ of the classifying space BG [**C-F**]. If S is a Sylow p-subgroup of G, the inclusion map induces a splitting epimorphism $MU_*(BS)_{(p)} \to MU_*(BG)_{(p)}$. Hence we need to know first $MU_*(BG)$ for p-groups G. Moreover the Quillen isomorphism $MU_*(-)_{(p)} \cong MU_{*(p)} \otimes_{BP*} BP_*(-)$ shows that we need to know only $BP_*(BG)$.

When G is a cyclic or quaternion group, the graded module associated to the dimensional filtration gr $BP_*(BG)$ is isomorphic to $BP_* \otimes H_*(BG)$ since $H_{even}(BG) \cong 0$ [**M**]. By Johnson-Wilson [**J-W**], gr $BP_*(BG)$ is given for an elementary abelian p-group using arguments to generalize Künneth formula. In this paper we determine BP_*-module structure of $BP_*(BG) \bmod (p, v_1, \cdots)^2$ for nonabelian groups of the order p^3. For $p=2$, the new group is the dihedral group D_4. The bordism group $BP_*(BD_{2q})$, q: prime$\neq 2$, was studied by Kamata-Minami [**K-M**] in early seventies.

Recall the Milnor primitive operation $Q_0=\beta$, $Q_1=p^1\beta-\beta p^1$ $(=Sq^2Sq^1-Sq^1Sq^2$ for $p=2)$. For the above groups, we can extend the operation Q_1 on $H^*(BG)$ so that $Q_1|H^{even}(BG)=0$. Let us write by $H(-;Q_1)$ the homology with the differential Q_1. Then we know (compare [**T-Y**])

$$\mathrm{gr}\, BP^*(BG) \cong BP^* \otimes H(H^*(BG);Q_1) \oplus BP^*/(p, v_1) \otimes \mathrm{Im}\, Q_1$$

since $d_{2p-1}=v_1 \otimes Q_1$ is the only non zero differential in the Atiyah-Hirzebruch spectral sequence. The similar fact occurs for the BP_*-homology

$$\mathrm{gr}\, BP_*(BG) \cong BP_* s^{-1} H(H^*(BG);Q_1) \oplus BP_*/(p, v_1) s^{-1} H^{odd}(BG)$$

where s^{-1} is the shift map which decreases degree by one. Here we use the spectral sequence $E_2^{*,*}=\mathrm{Ext}_{BP*}(BP^*(BG), BP^*) \Rightarrow BP_*(BG)$. In particular, generators and relations are given explicitely for $BP_*(BD_4)$ in the last section.