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On the semisimplicity of Hecke algebras

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0. Let (W, S) be a Coxeter system [**2**], t an indeterminate, $q=t^2$, and $H(W, t)$ a free $\boldsymbol{C}[t]$-module with a basis $\{T(w)\}_{w\in W}$ parametrized by the elements of W. Here $\boldsymbol{C}$ denotes the field of complex numbers. Then $H(W, t)$ has an associative $\boldsymbol{C}[t]$-algebra structure characterized by the conditions

$$(T(s)+1)(T(s)-q)=0, \qquad \text{if } s\in S,$$

and

$$T(w)T(w')=T(ww'), \qquad \text{if } l(w)+l(w')=l(ww'),$$

where l is the length function [**2**]. See [**2**; Chap. 4, § 2, Ex. 23] for the algebra structure of $H(W, t)$. See [**5**] for the significance of $H(W, t)$ in the representation theory. Let α be a complex number, $\varphi_\alpha: \boldsymbol{C}[t]\to\boldsymbol{C}$ the $\boldsymbol{C}$-algebra homomorphism defined by $\varphi_\alpha(t)=\alpha$, and $H(W, \alpha)=H(W, t)\otimes_{\boldsymbol{C}[t]}(\boldsymbol{C}, \varphi_\alpha)$.

From now on, we assume that W is finite, and (except in the final remark) not of type $A_1\times\cdots\times A_1$. Let w_0 be the longest element of W, $N=l(w_0)$, and $G(q)=q^N\sum_{w\in W}q^{l(w)}$.

The purpose of this note is to prove the following theorem.

THEOREM. *The $\boldsymbol{C}$-algebra $H(W, \alpha)$ is semisimple if and only if $G(\alpha^2)\neq 0$.*

1. Let

$$R_i: H(W, t)\longrightarrow M_{n_i\times n_i}(\boldsymbol{C}[t]) \qquad (i=1, 2)$$

be $\boldsymbol{C}[t]$-algebra homomorphisms. Here $M_{m\times n}$ denotes the set of $m\times n$-matrices. Let $T\hat{}(w)=q^{N-l(w)}T(w^{-1})$, $A\in M_{n_1\times n_2}(\boldsymbol{C}[t])$ and

$$B=\sum_{w\in W}R_1(T(w))AR_2(T\hat{}(w)).$$

LEMMA. *For $x\in W$, $R_1(T(x))B=BR_2(T(x))$.*

PROOF. We may assume that $x=s\in S$. Let $X=\{w\in W \mid l(sw)>l(w)\}$. Since W is a disjoint union of X and sX, it is enough to prove that

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