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Hamiltonian circuits on simple 3-polytopes with up to 30 vertices

By Haruko OKAMURA

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§1. Introduction.

Klee in [5] asked what is the minimun number, n, of vertices for a simple 3-polytope with no Hamiltonian circuit, that is, no closed path on the edges of the polytope which goes through each vertex exactly once. The smallest known non-Hamiltonian simple 3-polytope has 38 vertices (see p. 359 in [5]), so $n \leqq 38$. Lederberg [6] proved $n \geqq 20$, Butler [2] and Goodey [4] proved $n \geqq 24$, Barnette and Wegner [1] proved $n \geqq 28$. In this paper we show $n \geqq 32$.

THEOREM. *Every simple* 3-*polytope of order* 30 *or less is Hamiltonian.*

By Steinitz's theorem [5, p. 235] a graph is the graph of a simple 3-polytope if and only if it is planar, 3-connected and 3-valent. A set S of edges of a graph is called a cut if the removal of these edges separates G into two connected components and no proper subset of S has this property. If the cardinality of the cut is k it will be called a k-cut. The components separated by a k-cut are called k-pieces. A cut will be called non-trivial if each of its k-pieces contains a circuit, trivial otherwise. A non-trivial k-cut will be called essential if each of its k-pieces contains more than k vertices, non-essential otherwise. A graph will be called cyclically k-connected if every l-cut with $l<k$ is trivial, it will be called cyclically exactly k-connected if it is cyclically k-connected but not cyclically $(k+1)$-connected. The order of a graph G will be denoted by $|G|$.

§2. Preliminaries.

In this section we prepare some lemmas. By [2] and [4] we have Lemma 1.

LEMMA 1. *In any simple* 3-*polytope of order* 22 *or less each edge is used by some Hamiltonian circuit.*

By [3] we have Lemma 2.

LEMMA 2. *Any minimal non-Hamiltonian simple* 3-*polytope of order* 34 *or less is cyclically exactly* 4-*connected and has no essential* 4-*cut.*