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On a conjecture of Nakai on Picard principle

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A nonnegative locally Hölder continuous function $P(z)$ on the punctured closed unit disk $0<|z|\leqq 1$ will be referred to as a *density* on Ω : $0<|z|<1$. We view Ω as interior of the bordered surface: $0<|z|\leqq 1$; so we consider the boundary $\partial\Omega$ to be the circle: $|z|=1$. The *elliptic dimension* of a density P on Ω at $z=0$, $\dim P$ in notation, is defined to be the dimension of the half module $\mathcal{P}_P$ of nonnegative solutions of the equation $\Delta u=Pu$ $(\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2)$ on Ω with vanishing boundary values on $\partial\Omega$: $|z|=1$. After Bouligand we say that the *Picard principle* is valid for P at $z=0$ if $\dim P=1$. We are interested in determining those densities P on Ω for which the Picard principle is valid. We observe the following example of Nakai [**26, 29**]: Let $P_\mu(z)=|z|^{-\mu}$ and $P_{-\infty}(z)=0$. Then

$$(1) \qquad \dim P_\mu=\begin{cases} 1 & (\mu\in[-\infty, 2]) \\ \mathfrak{c} & (\mu\in(2, +\infty)) \end{cases}$$

where $\mathfrak{c}$ is the cardinal number of continuum. In this connection Nakai *conjectured* that the Picard principle is valid for general densities $P(z)$ on Ω with $P(z)=O(|z|^{-2})$ $(z\to 0)$. The *purpose* of this paper is to prove this conjecture in the *affirmative*. Namely we shall prove the following

MAIN THEOREM. *The Picard principle is valid for any density $P(z)$ on Ω with $P(z)=O(|z|^{-2})$ $(z\to 0)$.*

The proof of this theorem will be given in Section 1. A formulation of the Harnack inequality by S. Itô [**11**] will play an essential role in our proof. The author is grateful to Professor Itô for his advice on this inequality. In Section 2 we will discuss the order of the generator $g(z)$ of $\mathcal{P}_P$ as $z\to 0$ for $P(z)=O(|z|^{-2})$ $(z\to 0)$. We will establish the following inequality:

$$(2) \qquad C_1\log\frac{1}{|z|}\leqq g(z)\leqq C_2|z|^{-c}$$

on Ω where C_1 and C_2 are positive constants and $c=\sup_\Omega|z|P(z)^{1/2}$. In the final Section 3 we will mention two generalizations of the main theorem. We will show that the condition $P(z)\leqq\text{const.}\,|z|^{-2}$ on Ω in the main theorem can be relaxed to $P(z)\leqq\text{const.}\,|z|^{-2}$ only on a sequence of disjoint concentric annuli A_n

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