J. Math. Soc. Japan
Vol. 23, No. 4, 1971

On some dissipative boundary value problems for the Laplacian

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(Received Feb. 26, 1971)
(Revised June 18, 1971)

§ 1. Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^{n+1}$ with boundary Γ of class C^∞. $\bar{\Omega}=\Omega\cup\Gamma$ is a C^∞-manifold with boundary. For a function u in $C^\infty(\bar{\Omega})$ and $s\in\boldsymbol{R}$, $\|u\|_s$ denotes Sobolev norm of u of order s.

We consider Laplacian $\Delta=\dfrac{\partial^2}{\partial x_1{}^2}+\cdots+\dfrac{\partial^2}{\partial x_{n+1}{}^2}$ in Ω together with the homogeneous boundary condition

$$\mathscr{B}u\Big|_\Gamma=\frac{\partial u}{\partial\nu}+(a+ib)u+(a_0+ib_0)u\Big|_\Gamma=0,$$

where ν is the unit exterior normal to Γ, a and b are real C^∞-vector fields on Γ and a_0 and b_0 are real C^∞-functions on Γ.

The following problem is still open: "Characterize those couples $(\Omega, \mathscr{B})$ for which there exists a constant C such that the estimate

$$-\mathrm{Re}\,(\Delta u, u)+C\|u\|_0{}^2\geqq 0 \tag{1.1}$$

holds for any u in $C^2(\bar{\Omega})$ satisfying $\mathscr{B}u|_\Gamma=0$.

A well-known necessary condition for the estimate (1.1) to hold is that

$$|b(x)|\leqq 1, \tag{1.2}$$

where $|b(x)|$ is the length of the vector $b(x)$, $x\in\Gamma$ ([**6**]). On the other hand if $|b(x)|<1$ at every point $x\in\Gamma$, then there exist constants $C_0>0$ and C_1 such that the estimate

$$-\mathrm{Re}\,(\Delta u, u)+C_1\|u\|_0{}^2\geqq C_0\|u\|_1{}^2 \tag{1.3}$$

holds for any u in $C^2(\bar{\Omega})$ satisfying $\mathscr{B}u|_\Gamma=0$. (J. L. Lions [**8**], see also [**1**], [**6**], [**10**].)

In this note assuming (1.2), we are concerned with the following estimate:

Main part of this work was done when D. Fujiwara was staying at the Courant Institute of Mathematical Sciences, New York University, with the National Science Foundation, Grant NSF-GP-19617.