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Remarks on pseudo-differential operators*

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§ 0. Introduction.

In a recent paper [2] Hörmander defined pseudo-differential operators through a function class $S^m_{\rho,\delta}(\Omega)$, $0 \leqq \delta$, $0 < \rho$, for an open set Ω in R^n. We say $p(x;\xi) \in S^m_{\rho,\delta}(\Omega)$, when $p(x;\xi)$ belongs to C^∞ $(R^n \times R^n)$ and, for every compact set $K \subset \Omega$ and all α, β, there exist constants $C_{\alpha,\beta,K}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta p(x;\xi)| \leqq C_{\alpha,\beta,K}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}, \qquad x \in K, \quad \xi \in R^n,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ are multi-indices whose elements are non-negative integers and

$$\partial_{x_j} = \frac{\partial}{\partial x_j}, \qquad \partial_{\xi_j} = \frac{\partial}{\partial \xi_j}, \qquad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \qquad \partial_\xi^\beta = \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_n}^{\beta_n},$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad |\beta| = \beta_1 + \cdots + \beta_n.$$

In the present paper we shall study the H_s theory of pseudo-differential operators for the special case: $0 \leqq \delta < \rho \leqq 1$, $\Omega = R^n$ and $C_{\alpha,\beta,K} = C_{\alpha,\beta}$ (independent of K). In this case Hörmander [2] proved an inequality of the form

$$\|p(X;D_x)u\|_0 \leqq C_p \|u\|_0,$$

when $m=0$, and Lax-Nirenberg [7] proved a sharp form of Gårding's inequality:

$$\mathcal{R}e\,(\boldsymbol{p}(X;D_x)\boldsymbol{u}, \boldsymbol{u}) \geqq -K\|\boldsymbol{u}\|_0^2,$$

when $m=1$, $\rho=1$ and $\delta=0$. But we must remark here that it is complicated to derive the corresponding inequalities when m is an arbitrary real number and the $\|\cdot\|_0$ norm is replaced by the $\|\cdot\|_s$ norm for real s. In the present note the space $\mathcal{B}$, i.e., the set of C^∞ functions in R^n (or $R^n \times R^n$) whose derivatives are all bounded, plays an important role.

In Section 1 we define the operator class $\mathcal{S}^m_{\rho,\delta}$, $0 \leqq \delta < \rho \leqq 1$, and, through it, the class $\mathcal{L}^m_{\rho,\delta}$ of pseudo-differential operators. The main theorems, which

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