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A note on a theorem of S. Kurepa

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Let E^n be the n-dimensional Euclidean space, which we interpret also as a vector space in the usual manner. Given a point $c \in E^n$ and a positive number ρ, we shall denote by $K(c, \rho)$ the closed sphere of centre c and radius ρ, i. e. the set of the points x such that $|x-c| \leqq \rho$.

In his illuminating paper [**1**], S. Kurepa has recently proved the following result:

THEOREM. *Let P be a measurable set in E^n, and $\alpha_1, \alpha_2, \cdots, \alpha_t$ any finite sequence of real numbers such that*

$$0<|\alpha_k| \leqq 1 \quad for \quad k=1, 2, \cdots, t.$$

If x_0 is a point of density for the set P, there exist two concentric spheres $K(x_0, r)$ and $K(x_0, r')$ of centre x_0 such that $r \geqq r'$ and that with each point x of the smaller sphere $K(x_0, r')$ we can associate a non-vanishing vector $h(x) \in E^n$ so as to fulfil

$$x+\alpha_k h(x) \in K(x_0, r) \cap P \quad for \quad k=1, 2, \cdots, t.$$

It may be remarked that the conditions $|\alpha_k| \leqq 1$ above are only expediential. The assertion clearly remains valid for any finite sequence of real numbers $\alpha_1, \alpha_2, \cdots, \alpha_t$ other than 0.

The proof of the theorem, as given by Kurepa, involves some properties of characteristic functions of sets. But the author has found out that the theorem admits of a simpler proof which does not use such a tool. To show this is the object of the present note. By the way, it will turn out in the course of our proof that the radii r and r' of the assertion can be so chosen that $r=2r'$.

NOTATIONS. (i) The Lebesgue measure of any measurable set X in E^n will be written $m(X)$. (ii) For any set $Y \subset E^n$ and any real number α, we shall denote by αY the set of the vectors αy where y ranges over Y; so that, if Y is in particular measurable, then $m(\alpha Y)=|\alpha|^n m(Y)$. (iii) If U and V are two sets in E, then by $U \setminus V$ we shall understand the set of the points of U which are not in V, and by $U-V$ the set of all the vectors $u-v$ where $u \in U$ and $v \in V$.