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On the theory of ordinal numbers, II.

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In his former paper [2], the author formalized a system of axioms for the theory of ordinal numbers, which will be denoted by Γ_0 in this paper. He proved in [2] that the consistency of the set theory follows from that of Γ_0. In this paper the proof for its converse is given, namely we shall show that Γ_0 is consistent, if the set theory is consistent.

For this purpose we shall give three formalizations Γ_1, Γ_2 and Γ_3 of set theory. Among them Γ_1 is the 'weakest' and the Γ_3 the 'strongest' one, and Gödel's axiom system A, B, C, D and E lies between Γ_1 and Γ_2.

More precisely, the system Γ_1 is the system obtained from Gödel's axiom system A, B, C, D and E by replacing any class variable by a well formed formula of Gentzen's *LK* (cf. [6]). The system Γ_2 is obtained again from Gödel's system by replacing any class variable by a variable for formulas. However, in Γ_2 we use the logic *HLC* [3], which means the predicate logic of the second order and the first level. The system Γ_3 is so constructed that it contains enough axioms for our purpose to reduce Γ_0 to Γ_3. In Γ_3 we use the logic *FLC* [2], which means the logic without variables for formulas but with bound variables for functions of any order.

We first prove the consistency of Γ_2 under the assumption of the consistency of Γ_1 (§2 of Chapter I). Then we show that Γ_3 is consistent, if Γ_2 is consistent (§3 of Chapter I). Finally, by using the restriction theory in the author's paper [4], we construct a model for Γ_0 in the set theory Γ_3 (§2 of Chapter II). Consequently the consistency of the ordinal number theory Γ_0 is proved, provided that the set theory Γ_1 is consistent (§2 of Chapter II).

Chapter I. Three formalizations of set theory.

§1. The first formalization.

We give the first formalization of set theory by the following axioms Γ_1 in *LK*.

1.1. $\forall x \forall y (\forall z (z \in x \mapsto z \in y) \mapsto x = y)$

1.2. $\forall x \forall y \forall z (z \in \{x, y\} \mapsto x = z \vee y = z)$

1.3. $\forall x \forall y (y \in \mathrm{U}(x) \mapsto \exists z (y \in z \wedge z \in x))$