Journal of the Mathematical Society of Japan Vol. 9, No. 3, July, 1957

Curvature and relative Betti numbers.

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(Received Feb. 18, 1957)
(Revised April 17, 1957)

Many interesting relations between curvature and Betti numbers in a compact orientable Riemannian space are obtained by S. Bochner, A. Lichnerowicz and K. Yano (See [1]). We shall generalize these results to the case of the domain with boundary. Three quadratic forms related to the curvature tensors of the domain and of the boundary have an intimate connection with the absolute and relative Betti numbers of the doman. Our results are the consequences of the theorems concerning harmonic forms and Betti numbers given by Conner [3], Duff and Spencer [4].

Let $\mathfrak{M}$ be an n-dimensional orientable Riemannian manifold of C^{∞} with positive definite metric $ds^2=g_{ij}dx^i dx^j$. Let $\mathfrak{D}$ be an open set in $\mathfrak{M}$ with $(n-1)$-dimensional boundary $\mathfrak{B}$ of C^{∞} and $\tilde{\mathfrak{D}}$ be an open set containing $\mathfrak{D}$ and $\mathfrak{B}$. We assume that $\mathfrak{D}\cup\mathfrak{B}$ is compact. Local coordinates in $\tilde{\mathfrak{D}}$ and in $\mathfrak{B}$ are denoted by x^i $(i=1, 2, \cdots, n)$ and u^{λ} $(\lambda=2, 3, \cdots, n)$ respectively. Let N^i be the components of the unit contravariant outward normal vector to the boundary $\mathfrak{B}$ and we put $X_{\lambda}^{i}=\dfrac{\partial x^i}{\partial u^{\lambda}}$.

The local coordinates x^i in $\tilde{\mathfrak{D}}$ and u^{λ} in $\mathfrak{B}$ are both oriented in positive sense, so that we have

$$\delta_{i_1 i_2\cdots i_n} N^{i_1} X_{\lambda_2}^{i_2}\cdots X_{\lambda_n}^{i_n}\bar{\delta}^{\lambda_2\cdots\lambda_n}>0\,, \tag{1}$$

where $\delta_{i_1\cdots i_n}$ and $\bar{\delta}^{\lambda_2\cdots\lambda_n}$ are Kronecker deltas in $\tilde{\mathfrak{D}}$ and in $\mathfrak{B}$. Let D_i denote the covariant differential operators with respect to the metric form $g_{ij}dx^i dx^j$ in $\tilde{\mathfrak{D}}$ and $\varepsilon_{i_1\cdots i_n}=\sqrt{g}\,\delta_{i_1\cdots i_n}$.

We shall adopt the following notations in accordance with the previous paper [2]. Let $A_{(p)}$ and $B_{(q)}$ be anti-symmetric covariant tensors of order p and q defined in $\tilde{\mathfrak{D}}$ and of C^{∞}.