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# Conformally Flat Riemann Spaces of Clase One

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When an $n(\geqq 3)$-dimensional Riemann space $C_n$ with a metric defined by a positive-definite quadratic differential form is conformal to a flat space and is of class one, at least $(n-1)$ principal normal curvatures are equal. If all of them are equal, $C_n$ is of constant curvature.

This theorem was proved by J. A. Schouten,[1] only when $(n+1)$-dimensional flat $E_{n+1}$ enclosing $C_n$ is euclidean. But, even if $E_{n+1}$ is not euclidean, we can prove it similarly as follows.

Since $C_n$ is conformally flat, the curvature tensor is written in the from

$$(0\cdot1)\qquad R_{hijk}=g_{hj}l_{ik}+g_{ik}l_{hj}-g_{hk}l_{ij}-g_{ij}l_{hk}\,;$$

where we put

$$(0\cdot2)\qquad l_{ij}=\frac{1}{n-2}\left(R_{ij}-\frac{R}{2(n-1)}g_{ij}\right).$$

If $C_n$ is of class one, the Gauss equation

$$(0\cdot3)\qquad R_{hijk}=e(H_{hj}H_{ik}-H_{hk}H_{ij})\qquad (e=\pm1)$$

is satisfied; where $e=+1$ if enclosing $E_{n+1}$ is euclidean. Referring to the coordinate system $(C)$ such that coordinate lines are lines of curvature, we have from $(0\cdot1)$ and $(0\cdot3)$,

$$(0\cdot4)\qquad l_{ii}+l_{jj}=eH_{ii}H_{jj}\qquad (i\neq j)\,.$$

Making use of $(0\cdot4)$, we can prove easily the above theorem.

In this paper we look for the condition that $C_n(n>3)$ not of constant curvature be of class one. It is very difficult to express the resultant system[2] explicitly as in the paper of T. Y. Thomas for general Riemann spaces of class one. C. B. Allendoerfer could avoid this difficulty for the Einstein spaces.[3] We shall obtain an analogous result for conformally flat spaces as follows.

**I.** We have from $(0\cdot3)$

$$(1\cdot1)\qquad H_{lm}R_{hijk}-H_{lk}R_{hijm}-H_{jh}R_{mkli}+H_{ji}R_{mklh}=0\,.^{(4)}$$