

On ruled surfaces of genus 1

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(Received Aug. 24, 1968)

In this paper we study the complex structure of ruled surfaces of genus 1—complex analytic projective line bundles over non-singular elliptic curves. The classification of these bundles were given earlier by Atiyah [1]. In Section 1 of the present paper we make further classification of them as complex analytic surfaces. The underlying topological (or differentiable) manifold of a ruled surface of genus 1 is an S^2 -bundle over $S^1 \times S^1$, where S^2 and S^1 denote, respectively, a 2-sphere and a circle. We prove, in Section 2, that such bundles have two types: a trivial bundle $E_0 = S^1 \times S^1 \times S^2$ and a non-trivial bundle E_1 , and that if a surface S is topologically (or differentiably) homeomorphic to E_0 or E_1 , then S is a ruled surface of genus 1 (Theorem 2). Combining this with the result of Section 1, we can determine all the complex structures on E_0 and E_1 . We note that, while the set of all the complex structures on E_0 forms a continuum, E_1 admits only a countable number of complex structures. In Section 3 we give explicit construction of the complex analytic families of the above complex structures of which the existence is asserted by a theorem of Kodaira-Nirenberg-Spencer [10]. In those families we see the “jump” phenomenon of complex structures, which is characteristic to ruled surfaces.

F. Enriques ([4]) first discovered that, if an algebraic surface S has the numerical characters: $p_g = c_1^2 = 0$ and $q = 1$, then S is either a ruled surface (of genus 1) or an elliptic surface, where p_g , q and c_1 denote, respectively, the geometric genus, the irregularity and the first Chern class of S . In Section 4 we examine those surfaces which are both ruled and elliptic, in other words, we find the ruled surfaces which have another fibering of elliptic curves. A similar method used in proving Theorem 5 is applicable to the explicit determination of the structure of so called (irregular) hyperelliptic surfaces (Enriques-Severi [5]).

§ 1. Biholomorphic classification of ruled surfaces of genus 1.

By a surface we shall mean a connected compact complex manifold of complex dimension 2. We shall follow the notation and terminology of Kodaira