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ON MARCINKIEWICZ INTEGRAL

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1. Introduction. Let P be a closed set in R^n and $\delta(x) = \delta_P(x)$ denote the distance of the point x from P. Let λ be a positive number and $f \in L^p(R^n)$, $1 \leqq p \leqq \infty$. We shall call the integral

$$J_\lambda(x) = J_\lambda(x; f) = \int_{R^n} \frac{\delta^\lambda(y) f(y)}{|x - y|^{n+\lambda}} dy \tag{1.1}$$

to be *the Marcinkiewicz "distance function" integral* of f.

Concerning this integral, following results are known:

If $f \in L^1(R^n)$, then the integral (1.1) *converges almost everywhere in P. In particular, if P is bounded and is contained in a finite cube Q, then*

$$\int_Q \frac{\delta^\lambda(y)}{|x - y|^{n+\lambda}} dy \tag{1.2}$$

is finite almost everywhere in P.

On the other hand, *if $|\complement P| < \infty$*[1], *then*

$$\int_{R^n} \frac{\delta^\lambda(y)}{|x - y|^{n+\lambda}} dy \tag{1.3}$$

is almost everywhere finite in P. For these results we refer the reader to Zygmund [7] and Stein [6; Chapter I].

The integral of the form (1.1) diverges in general outside P, so some variants are introduced, namely

$$H_\lambda(x) = \int_{R^n} \frac{\delta^\lambda(y) f(y)}{|x - y|^{n+\lambda} + \delta^{n+\lambda}(x)} dy \tag{1.4}$$

and

$$H'_\lambda(x) = \int_{R^n} \frac{\delta^\lambda(y) f(y)}{|x - y|^{n+\lambda} + \delta^{n+\lambda}(y)} dy \,. \tag{1.5}$$

In view of the relation $|\delta(x) - \delta(y)| \leqq |x - y|$ we have by Jensen's inequality

$$|x - y|^{n+\lambda} + \delta^{n+\lambda}(x) \approx |x - y|^{n+\lambda} + \delta^{n+\lambda}(y) \,,$$

[1] $\complement E$ is the complement of the set E and $|E|$ denotes the Lebesgue measure of E.