NOTE ON DIRICHLET SERIES (V) ON THE INTEGRAL FUNCTIONS DEFINED BY DIRICHLET SERIES (I)

CHUJI TANAKA

(Received March 23, 1953)

1. Introduction. Let us put

$$F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it,\ 0 \leqq \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_n \to +\infty). \tag{1.1}$$

Let (1.1) be uniformly convergent in the whole plane, i.e. for any given σ $(-\infty < \sigma < +\infty)$, (1.1) be uniformly convergent for $\sigma \leqq \Re(s)$. Then (1.1) defines the integral function, and for any given σ, $\operatorname*{Sup}_{-\infty<t<+\infty} |F(\sigma + it)|$ has the finite value $M(\sigma)$. After J. Ritt ([1], pp. 18–19) we can define the order and type of (1.1) as follows:

DEFINITION. *The order of* (1.1) *is defined by*

$$\rho = \overline{\lim_{\sigma\to-\infty}}(-\sigma)^{-1}\cdot \log^+\log^+ M(\sigma), \tag{1.2}$$

where $M(\sigma) = \operatorname*{Sup}_{-\infty<t<+\infty} |F(\sigma + it)|$, $\log^+ x = \operatorname{Max}(0, \log x)$. *If* $0 < \rho < +\infty$, *the type* k *of* (1.1) *is defined by*

$$k = \overline{\lim_{\sigma\to-\infty}} 1/\exp((-\sigma)\rho)\cdot \log^+ M(\sigma). \tag{1.3}$$

J. Ritt [2], S. Izumi [3] and K. Sugimura [4] have given formulas determining ρ and k in terms of $\{a_n\}$ $(n = 1, 2, \ldots.)$ under some additional conditions imposed upon $\{\lambda_n\}$ $(n = 1, 2, \ldots.)$. In this note, we shall establish more general formulas determining ρ and k in terms of $\{a_n\}$ $(n = 1, 2, \ldots.)$.

2. Theorem. The main theorem reads as follows:

MAIN THEOREM. *Let* (1.1) *be uniformly convergent in the whole plane. Then we have*

$$\overline{\lim_{x\to+\infty}}(x \log x)^{-1}\cdot \log T_x = -\rho_u^{-1}, \tag{2.1}$$

where

$$\left\{\begin{array}{ll} \text{(i)} & T_x = \operatorname*{Sup}_{-\infty<t<+\infty} \left| \sum_{[x]\leqq\lambda_n<x} a_n \exp(-i\lambda_n t) \right|^{*}, \\ \text{(ii)} & M_u(\sigma) = \operatorname*{Sup}_{\substack{-\infty<t<+\infty \\ 1\leqq k<+\infty}} \left| \sum_{n=1}^{k} a_n \exp(-\lambda_n(\sigma + it)) \right|, \\ \text{(iii)} & \rho_u = \overline{\lim_{\sigma\to-\infty}}(-\sigma)^{-1}\cdot \log^+\log^+ M_u(\sigma) \qquad (\geqq 0). \end{array}\right.$$

---

* $[x]$ means the greatest integer contained in x.