ON INTEGRAL INEQUALITIES AND CERTAIN APPLICATIONS TO FOURIER SERIES

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1. **Introduction.** Let $\varphi(z)$ be a function regular in the unit circle, then Littlewood-Paley [4] introduced the following functions:

$$g(\theta, \varphi) = g(\theta) = \left\{ \int_0^1 (1-\rho)|\varphi'(z)|^2 d\rho \right\}^{1/2}, \quad z = \rho e^{i\theta}, \tag{1.01}$$

$$g_p(\theta, \varphi) = g_p(\theta) = \left\{ \int_0^1 (1-\rho)^{p-1}|\varphi'(z)|^p \, d\rho \right\}^{1/p}, \quad \text{for } p > 1 \tag{1.02}$$

and

$$g^*(\theta, \varphi) = g^*(\theta) = \left\{ \int_0^1 (1-\rho)\chi^2(\rho, \theta)\, d\rho \right\}^{1/2}, \tag{1.03}$$

where

$$\chi(\rho, \theta) = \left\{ \frac{1}{\pi} \int_0^{2\pi} |\varphi'(\rho e^{i\theta+it})|^2 P(\rho, t)\, dt \right\}^{1/2}, \tag{1.04}$$

and

$$P(\rho, t) = (1-\rho^2)/(1 - 2\rho\cos t + \rho^2). \tag{1.05}$$

They established important theorems in the theory of Fourier series by the use of the above integrals. These auxiliarly functions have been researched by other authors, A. Zygmund [7, 8], J. Marcinkiewicz-A. Zygmund [5], and G. Sunouchi [6], and they have given complete generalized forms and simple proofs.

If for some $p > 0$, the integral

$$\int_{-\pi}^{\pi} |\varphi(\rho e^{i\theta})|^p \, d\theta$$

remains bounded when $\rho \to 1-0$, the function $\varphi(z)$ is said to belong to the class H^p, then their theorems read as follows:

THEOREM A. *Let $\varphi(z) \in H^r$, $r > 0$, then we have*

$$\left\{ \int_0^{2\pi} g^r(\theta)\, d\theta \right\}^{1/r} \leqq A_r \left\{ \int_0^{2\pi} |\varphi(e^{i\theta})|^r \, d\theta \right\}^{1/r}, \tag{1.06}$$

where

$$A = O(r) \text{ as } r \to \infty \tag{1.07}$$