ON FRACTIONAL INTEGRATION

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1. Introduction. The present paper is devoted to give certain results for fractional integration which are related to the work of I. I. Hirschman, Jr.

Let $u(\theta)$ be a function in the class $L^p(0, 2\pi), p \geqq 1$ with mean value zero and its Fourier series be

$$u(\theta) \sim \sum' a_n e^{in\theta}$$

where $-\infty < n < \infty$ and $n \neq 0$.

The fractional integral $u_\alpha(\theta)$ of $u(\theta)$ of order α is defined by

$$u_\alpha(\theta) \sim \sum' a_n (in)^{-\alpha} e^{in\theta}$$

and let the Abel mean of $u(\theta)$ and $u_\alpha(\theta)$ be

$$u(r, \theta) = \sum' a_n r^{|n|} e^{in\theta}$$

$$u_\alpha(r, \theta) = \sum' a_n (in)^{-\alpha} r^{|n|} e^{in\theta}$$

We consider the following functions, the first due to I. I. Hirschman, Jr. [1] and the remains to G. Sunouchi [3],

$$g(\alpha ; \theta) = \left\{ \int_0^1 (1-r)^{1-2\alpha} |u_{\alpha-1}(r, \theta)|^2 dr \right\}^{1/2}$$

$$g^*(\alpha, \beta ; \theta) = \left\{ \int_0^1 (1-r)^{2(\beta-\alpha)} dr \int_0^{2\pi} \frac{|u_{\alpha-1}(r, \theta+t)|^2}{|1-re^{it}|^{2\beta}} dt \right\}^{1/2}$$

$$\delta(\alpha, k ; \theta) = \left\{ \int_0^{2\pi} |\Delta_{t(k)}^{k+1} u_\alpha(\theta)|^2 t^{-2\alpha-1} dt \right\}^{1/2},$$

where

$$\Delta_t^1 u_\alpha(\theta) = u_\alpha(\theta+t) - u_\alpha(\theta-t)$$

$$\Delta_t^{k+1} u_\alpha(\theta) = \Delta_t^1(\Delta_t^k u_\alpha(\theta))$$

$$t(k) = t/2(k+1).$$

The main purpose of this paper is to prove the following:

THEOREM 1. *Let $u(\theta) \in L^p(0, 2\pi)$, $p > 1$, then we have*

$$\|\delta(\alpha, k;\ \theta)\|_p \leqq A_p \|u\|_p$$

where $0 < \alpha < k+1\ (2 < p < \infty)$, $2/p - 1 < \alpha < k+1\ (1 < p < 2)$ and k is a positive integer or zero.

The constant A_p depends only on p, and not on the function $u(\theta)$.