CESÀRO SUMMABILITY OF WALSH-FOURIER SERIES

SHIGEKI YANO

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1. It is well known that the trigonometrical Fourier series of an integrable function is (C, α) $(\alpha > 0)$ summable almost everywhere. Moreover, the maximal theorems for (C, α) means of the Fouier series are known (see, for example, [8; § 10.22, p. 248]).

Recently, N. J. Fine has proved that the Walsh-Fourier series of an integrable function is (C, α) $(\alpha > 0)$ summable almost everywhere. In this note, we prove that the maximal theorems for the (C, α) means of Walsh-Fourier series are also true. For functions in L^p, $p > 1$, proofs are given by Paley [4] and Sunouchi [5]. Our proof is completely different from their ones, and is based on the estimation for (C, α) kernels of Walsh functions and a lemma of Fine [3]. For notations and background materials, the reader is referred to the paper of Fine [2].

THEOREM. *Let* $\sigma_n^{(\alpha)}(x) = \sigma_n^{(\alpha)}(x; f)$ *denote the* (C, α) *mean of the Walsh-Fourier series of an integrable function* $f(x)$. *Then for* $\alpha > 0$

$$\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)|^p \, dx \leqq A_{p,\alpha} \int_0^1 |f(x)|^p \, dx, \qquad p > 1, \tag{1}$$

$$\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)| \, dx \leqq A_\alpha \int_0^1 |f(x)| \log^+ |f(x)| \, dx + B_\alpha, \tag{2}$$

$$\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)|^r \, dx \leqq A_r \left\{ \int_0^1 |f(x)| \, dx \right\}^r, \qquad 0 < r < 1, \tag{3}$$

where the constants A, B *with the subscripts are dependent only on the quantities indicated by subscripts.*

For the proof of the theorem, we need the following lemma;

LEMMA. *Let* E *be a measurable subset of the interval* $[0, 1]$, $D(x) = \rho(x, E)$, *the distance from* x *to* E, *and* $\{h_n\}$, $1 \geqq h_0 \geqq h_1 \geqq h_2 \geqq \ldots \geqq 0$, *be a sequence satisfying*

$$\sum_{h_j \leqq \delta} h_j \leqq M\delta,$$

$$\sum_{h_j > \delta} \frac{1}{h_j} \leqq \frac{M}{\delta}$$

for a constant M *and for every* $\delta > 0$. *Let us set*