ON THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION III[1)]

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(Received January 26, 1961)

1. Introduction. Let $f(x)$ be integrable $(-\pi, \pi)$ and be periodic with period 2π, and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

We denote the Riesz typical means of the above series by

$$X_n^k(x) = \sum_{\nu=0}^{n} A_\nu(x)(1 - \nu^k/n^k),$$

then the following results are known [A. Zygmund [6] and G. Sunouchi-C. Watari [5]).

(1°) $|f(x) - X_n^k(x)| = o(n^{-k})$ uniformly, implies that $f(x)$ is a constant.

(2°) $|f(x) - X_n^k(x)| = O(n^{-k})$ uniformly, implies

$|f^{(k)}(x)| \leqq M$ (when k is an even integer)

$|\tilde{f}^{(k)}(x)| \leqq M$ (when k is an odd integer).

(3°) If $|f^{(k)}(x)| \leqq M$ (when k is an even integer)

$|\tilde{f}^{(k)}(x)| \leqq M$ (when k is an odd integer),

then

$$|f(x) - X_n^k(x)| = O(n^{-k}) \text{ uniformly.}$$

We denote the Riesz means of the α-th[2)] order of the Fourier series of $f(x)$ by

$$X_n^{k,(\alpha)}(x) = \sum_{\nu=0}^{n} A_\nu(x)(1 - \nu^k/n^k)^\alpha,$$

then we have proved the same results. In fact, the propositions (1°) and (2°)

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1) Research supported in part by the National Science Foundation (U.S.A.).

2) We assume α is a positive integer. $X_n^{k,(\alpha)}(x)$ is different from ordinary Riesz means which have a continuous parameter n. But (C, α)-summability implies $X_n^{k,(\alpha)}$-summability. See M. Riesz [3].